Primality Statistics

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Progress on prime gaps

Conjecture. (Polignac, 1849) *For each integer* $k \geq 1$ *there are infinitely many prime pairs differing by* $2k$*.*

Theorem. (Zhang, 2013) *There are infinitely many prime pairs differing by at most* 70,000,000*.*

Here is Zhang lecturing on his theorem at Harvard.

Work of Maynard, Tao, and others reduced bound to at most 246.

What Ross Program problem sets used to look like

OSU Number Theory Problem Set #6 A.E. Ross Columbus 6/29/86

Reasoning.

P1. We propose that we should accept the following properties as a description of $\mathbb{Z}$: We assume that $\mathbb{Z}$ is a commutative ring with identity and cancellation law, that $\mathbb{Z}$ is totally ordered under "$\leq$" and that its positive elements are well-ordered. Do these properties of $\mathbb{Z}$ suffice to resolve all the problems concerning $\mathbb{Z}$ with which we have dealt to date?

Technique of generalization.

P2. Does some form of division algorithm which leads to Euclid's algorithm exist in $\mathbb{Z}[i]$?

Prove or disprove and salvage if possible.

P3. If a, b, c are in $\mathbb{Z}$, then the Diophantine equation $ax + by = c$ has at least one solution in integers x, y.

P4. U_m has exactly $\varphi(m)$ elements.

P5. U_m is a group under multiplication in $\mathbb{Z}_m$.

P6. Use Euclid's algorithm to calculate the g.c.d. of two successive Fibonacci numbers F_m, F_{m+1}. What is the number of steps required? Explain.

P7. $n \in \mathbb{Z} \Rightarrow \prod_{\substack{d \mid n \\ d > 0}} d = n^{\tau(n)}$.

P8. $a \in U_m$, n the order of $a \Rightarrow n \mid \varphi(m)$.

P9. $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0 \in \mathbb{Z}_m[X]$ and $g(x) = b_\ell x^\ell + b_{\ell-1}x^{\ell-1} + \dots + b_0 \in \mathbb{Z}_m[X]$ and the degrees of $f(x)$ and $g(x)$ are n and ℓ respectively $\Rightarrow f(x) \cdot g(x)$ is of degree $n + \ell$.

P10. There exist infinitely many distinct positive primes in $\mathbb{Z}$. (May assume unique factorization into primes.)

P11. If $u_1 \in U_m$ has the order n_1 and $u_2 \in U_m$ has the order n_2, then the order of $u_1 u_2$ is $n_1 \cdot n_2$.

P12. $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_e^{\alpha_e}$ is the canonical decomposition of n into prime factors. Then
$$\sigma(n) = \prod_j \frac{p_j^{\alpha_j+1} - 1}{p_j - 1} \quad \text{and} \quad \tau(n) = \prod_{j=1} (\alpha_j + 1).$$

True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}_5[X]$.

What Ross Program problem sets used to look like

Numerical Problems (Some Food for Thought).

P13. The polynomial $2x^3+3x^2+x+4$ is written to base x. Write this polynomial to base $x-2$. (Compare with the method used in $\mathbb{Z}$)

P14. Calculate $\tau(15!)$. Explain.

P15. Find all the perfect squares in U_{29}.

Ingenuity.

P16. $m \in \mathbb{Z}$, $m>0$, $m \neq 3 \Rightarrow m^4+m^3+m^2+m+1$ is not a perfect square.

Exploration.

P17. Let $\frac{P_1}{Q_1}, \frac{P_2}{Q_2}, \frac{P_3}{Q_3}, \frac{P_4}{Q_4}, \ldots$ be the convergents of the simple continued fraction expansion of $\sqrt{7}$. Calculate the value of $P_j^2-7Q_j^2$ for $j=1,2,3,4,\ldots$. Calculate the value $P_j^2-dQ_j^2$ for the simple continued fraction expansion of $\sqrt{d}$ when $d=10$ and also when $d=13$. Do you care to make any conjectures about solving the Diophantine equation

$$x^2-dy^2=1, \quad d \neq \square .$$

P18. Problems 3 and 4 in the "Towards the Abstract" paper.

P19. Consider a polygon whose vertices are lattice points in the sense of P12, Set #2. Find a simple formula relating the area of such a polygon and the number of lattice points within and on the boundary of this polygon.

Getting quantitative

For $f_1(T), \ldots, f_r(T) \in \mathbf{Z}[T]$ fitting conditions of Hypothesis H, we set for $x > 1$

$$\pi_{f_1,\ldots,f_r}(x) = |\{n \leq x : f_1(n), \ldots, f_r(n) \text{ are all prime}\}|.$$

Example. Counting primes of the form $n^2 + 1$ uses

$$\pi_{T^2+1}(x) = |\{n \leq x : n^2 + 1 \text{ is prime}\}|.$$

What is $\pi_{T^2+1}(10)$?

Example. Counting twin primes uses

$$\pi_{T,T+2}(x) = |\{n \leq x : n \text{ and } n + 2 \text{ are prime}\}|.$$

What is $\pi_{T,T+2}(10)$?

Hypothesis H tells us when to expect $\pi_{f_1,\ldots,f_r}(x) \to \infty$ as $x \to \infty$. We want to refine this to say how quickly the count grows.

Getting quantitative

The approximations we seek for $\pi_{f_1,\dots,f_r}(x)$ will not be numerically close, but "order of growth" close.

Definition. For positive functions $f(x)$ and $g(x)$, write

$$f(x) \sim g(x)$$

if $f(x)/g(x) \to 1$ as $x \to \infty$. Say f is *asymptotic* to g.

That $f(x) \sim g(x)$ does not mean $f(x) - g(x)$ has to get small.

Example. $x^2 + x \sim x^2$

Example. $[x] \sim x$.

Example. $\log_b(3x) = \log_b 3 + \log_b x \sim \log_b x$ for each $b > 1$.

Example. $\displaystyle\sum_{n \le x} n = \frac{[x]([x]+1)}{2} \sim \frac{x^2}{2}$.

Exer: $f_1(x) \sim g_1(x), f_2(x) \sim g_2(x) \Rightarrow f_1(x)f_2(x) \sim g_1(x)g_2(x)$.
Exer: $f(x) \sim g(x) \Rightarrow 1/f(x) \sim 1/g(x)$.

The count of primes $\pi_T(x) = |\{n \leq x : n \text{ prime}\}|$ is usually written as $\pi(x)$, *e.g.*, $\pi(10) = |\{2, 3, 5, 7\}| = 4$.

Theorem. (Hadamard, de la Vallée Poussin, 1896) *As* $x \to \infty$,

$$\pi(x) \sim \frac{x}{\log x}.$$

Here and later, $\log x = \log_e x = \ln x$.

x	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	1229	9592	78498	664579	5761455
$x/\log x$	1085	8685	72382	620420	5428681
Ratio	1.1319	1.1043	1.0844	1.0711	1.0612

This was conjectured independently by Gauss and Legendre in the 1790s, so it took 100 years to prove the Prime Number Theorem.

Prime Number Theorem

Let p_n be the nth prime number. The Prime Number Theorem can be expressed in terms of the growth of p_n.

Theorem. *The Prime Number Theorem is equivalent to the asymptotic estimate*

$$p_n \sim n \log n.$$

n	10^4	10^5	10^6	10^7	10^8
p_n	104729	1299709	15485863	179424673	2038074743
$n \log n$	92103	1151293	13815511	161180957	1842068074
$\frac{p_n}{n \log n}$	1.1370	1.1289	1.1209	1.1131	1.1064

n	10^9	10^{10}	10^{11}	10^{12}	10^{13}	10^{14}
$\frac{p_n}{n \log n}$	1.1002	1.0948	1.0899	1.0855	1.0816	1.0781

Theorem. *The Prime Number Theorem is equivalent to the asymptotic estimate*

$$p_n \sim n \log n.$$

Proof. ($\Longrightarrow$) In the estimate $\pi(x) \sim x/\log x$, set $x = p_n$:

$$n \sim \frac{p_n}{\log p_n} \Longrightarrow \frac{n \log p_n}{p_n} \to 1 \Longrightarrow p_n \sim n \log p_n.$$

Take logarithms of both sides:

$$\log n + \log \log p_n - \log p_n \to 0 \Longrightarrow \frac{\log n}{\log p_n} + \frac{\log \log p_n}{\log p_n} - 1 \to 0.$$

Since $\log p_n \to \infty$ and $(\log x)/x \to 0$ as $x \to \infty$, the middle term goes to 0 in the limit and we're left with

$$\frac{\log n}{\log p_n} - 1 \to 0 \Longrightarrow \frac{\log n}{\log p_n} \to 1 \Longrightarrow \log p_n \sim \log n.$$

Therefore $p_n \sim n \log n$.

Prime Number Theorem

Theorem. *The Prime Number Theorem is equivalent to the asymptotic estimate*

$$p_n \sim n \log n.$$

Proof. ($\Longleftarrow$) From $p_n \sim n \log n$, taking logarithms as before implies $\log p_n \sim \log n$, so $n \sim p_n / \log n \sim p_n / \log p_n$. Thus

$$\pi(p_n) \sim \frac{p_n}{\log p_n}.$$

For $x \geq 2$, let $p_n \leq x < p_{n+1}$. Then $\pi(x) = n$ and

$$\frac{p_n}{n \log n} \leq \frac{x}{n \log n} = \frac{x}{\pi(x) \log \pi(x)} < \frac{p_{n+1}}{n \log n}.$$

Outer terms both tend to 1 since $n \log n \sim (n+1) \log(n+1)$, so

$$x \sim \pi(x) \log \pi(x).$$

Taking logarithms and arguing as before implies $\log x \sim \log \pi(x)$, so $x \sim \pi(x) \log x$. Thus $\pi(x) \sim x / \log x$.

Theorem. *As* $x \to \infty$,

$$|\{p \leq x : p \equiv 1 \bmod 4\}| \sim \frac{1}{2}\frac{x}{\log x}.$$

Let's translate this into a count of $\pi_{4T+1}(x)$:

$$\begin{aligned} |\{n \leq x : 4n+1 \text{ prime}\}| &= |\{\text{prime } p \leq 4x+1 : p \equiv 1 \bmod 4\}| \\ &\sim \frac{1}{2}\frac{4x+1}{\log(4x+1)} \\ &\sim \frac{1}{2}\frac{4x}{\log x} \\ &= 2\frac{x}{\log x}. \end{aligned}$$

For $(a, m) = 1$, $|\{p \leq x : p \equiv a \bmod m\}| \sim \frac{1}{\varphi(m)}\frac{x}{\log x}$. Same as

$$|\{n \leq x : mn + a \text{ prime}\}| \sim \frac{m}{\varphi(m)}\frac{x}{\log x} = \prod_{p|m}\frac{1}{1-1/p}\frac{x}{\log x}.$$

Just as $\pi(x) \sim x/\log x$ is equivalent to $p_n \sim n \log n$, the asymptotic estimate

$$\pi_{mT+a}(x) \sim \frac{m}{\varphi(m)} \frac{x}{\log x}$$

is equivalent to saying

$$n\text{th prime value of } a + mT \text{ is asymptotic to } \frac{\varphi(m)}{m} n \log n.$$

We want to adapt these theorems to count (conjecturally!)

$$\pi_f(x) = |\{n \leq x : f(n) \text{ is prime}\}|$$

asymptotically when $f(T)$ fits Bunyakovsky's conjecture.

We'll start by reviewing how Gauss conjectured the Prime Number Theorem from numerical data.

Density of primes near large numbers

Even before I had begun my more detailed investigations into higher arithmetic, one of my first projects was to turn my attention to the decreasing frequency of primes, to which end I counted primes in several chiliads. I soon recognized that behind all of its fluctuations, this frequency is on average inversely proportional to the logarithm. Gauss (letter to Encke, Dec. 1849)

n	$\lvert\{\text{primes in } [n, n+999]\}\rvert/1000$	$1/\log n$
10K	.106	.1085
50K	.089	.0924
100K	.081	.0868
500K	.079	.0762
1 M	.075	.0723
1.5 M	.083	.0703
2 M	.069	.0689
2.5 M	.064	.0678
3 M	.062	.0670

The basic probabilistic heuristic

Our prime heuristic will be

$$\text{Prob}(n \text{ prime}) = \frac{1}{\log n}.$$

Strictly speaking, this is problematic:

1. $1/\log 1 = \infty$ and $1/\log 2 > 1$.
2. Being prime is *not* a probabilistic concept.

Nevertheless, consider this more elementary heuristic:

$$\text{Prob}(n \text{ even}) = \frac{1}{2} \text{ and } \sum_{n \leq x} \text{Prob}(n \text{ even}) \sim \frac{x}{2} \sim |\{\text{even } n \leq x\}|. \checkmark$$

The expected number of primes up to x ("successes") should be a sum of probabilities over n up to x:

$$\pi(x) \overset{?}{\sim} \sum_{n \leq x} \text{Prob}(n \text{ prime}) = \sum_{2 \leq n \leq x} \frac{1}{\log n} \overset{!}{\sim} \frac{x}{\log x},$$

which **is** true. (The last asymptotic estimate will be used a lot.)

Data for Prime Number Theorem compared to a sum of probabilities

It is true that $\pi(x) \sim \dfrac{x}{\log x} \sim \displaystyle\sum_{2 \le n \le x} \frac{1}{\log n}$, but the last formula is the better asymptotic estimate for $\pi(x)$.

x	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	1229	9592	78498	664579	5761455
$x/\log x$	1085	8685	72382	620420	5428681
Ratio	1.1319	1.1043	1.0844	1.0711	1.0612
$\pi(x)$	1229	9592	78498	664579	5761455
$\displaystyle\sum_{2 \le n \le x} \frac{1}{\log n}$	1245	9629	78627	664918	5762209
Ratio	.9863	.9960	.9983	.9994	.9998

"For *verification within the limits of calculation*, [the formula used] is by no means indifferent and it will be found that it makes a vital difference in the plausibility of the results." Hardy & Littlewood

Primes of the form $n^2 + 1$

The prime heuristic suggests that

$$\pi_{T^2+1}(x) \overset{?}{\sim} \sum_{n \le x} \mathrm{Prob}(n^2 + 1 \text{ prime}) = \sum_{n \le x} \frac{1}{\log(n^2+1)} \overset{!}{\sim} \frac{x}{2 \log x}.$$

x	10^4	10^5	10^6	10^7	10^8
$\pi_{T^2+1}(x)$	841	6656	54110	456362	3954181
$\frac{x}{2\log x}$	543	4343	36191	310210	2714341
Ratio	1.5488	1.5325	1.4951	1.4711	1.4567
$\pi_{T^2+1}(x)$	841	6656	54110	456362	3954181
$\sum_{n \le x} \frac{1}{\log(n^2+1)}$	624	4816	39314	332460	2881105
Ratio	1.3471	1.3820	1.3763	1.3726	1.3724

How can we fix this?

Refining the prime heuristic

When we count how often $f(n)$ is prime, the idea of using

$$\text{Prob}(f(n) \text{ prime}) \sim \frac{1}{\log f(n)}$$

is too simple. We should **take into account** that the numbers are values of $f(T)$. This is related to conditional probability.

Example. $\text{Prob}(n \in \{1, \ldots, 100\}$ is a multiple of $4) = 1/4$, but $\text{Prob}(n \in \{1, \ldots, 100\}$ is a multiple of 4 **if** n is even$) = 1/2$.

Example. Numbers of the form $n^2 + 1$ for $n \geq 1$ are

$$2, 5, 10, 17, 26, 37, 50, 65, 82, 101, \ldots$$

and these alternate even/odd like all positive integers. They are *never* a multiple of 3, which makes it *more likely* such numbers are prime compared to random integers. They are *more often* a multiple of 5 (chances are 2 out of 5 instead of 1 out of 5), which makes it *less likely* they are prime compared to random integers.

Refining the prime heuristic

For prime p, the probability a random integer is not divisible by p is $1 - 1/p$. The probability a random $f(n)$ is not divisible by p is $1 - \omega_f(p)/p$, where

$$\omega_f(p) = |\{n \in \mathbf{Z}/(p) : f(n) \equiv 0 \bmod p\}| \leq p.$$

Note $f(T)$ fits Bunyakovsky condition iff $\omega_f(p) < p$ for all p.

New heuristic: $\text{Prob}(f(n) \text{ prime}) \sim \dfrac{C_f}{\log f(n)}$ where

$$C_f = \prod_p \frac{1 - \omega_f(p)/p}{1 - 1/p}.$$

1. $\omega_f(p) = 1$ doesn't change anything ($n^2 + 1$ and $p = 2$).
2. $\omega_f(p) = 0$ improves chance $f(n)$ is prime ($n^2 + 1$ and $p = 3$).
3. $\omega_f(p) > 1$ lessens chance $f(n)$ is prime ($n^2 + 1$ and $p = 5$).

Theorem. *An $f(T) \in \mathbf{Z}[T]$ fits Bunyakovsky condition iff $C_f \neq 0$.*

Using the new prime heuristic

If $f(T) = c_d T^d +$ terms of lower degree, and $c_d > 0$, then $f(n) \sim c_d n^d$, so $\log f(n) \sim \log(c_d n^d) \sim d \log n = (\deg f) \log n$. Therefore

$$\text{Prob}(f(n) \text{ prime}) \sim \frac{C_f}{\log f(n)} \sim \frac{C_f}{(\deg f) \log n},$$

so we change our guess for the growth of $\pi_f(x)$ to

$$\begin{aligned} \pi_f(x) &\sim \sum_{n \leq x} \text{Prob}(f(n) \text{ prime}) \\ &\sim \sum_{n \leq x} \frac{C_f}{(\deg f) \log n} \\ &\sim \frac{C_f}{\deg f} \frac{x}{\log x}, \quad \text{where } C_f = \prod_p \frac{1 - \omega_f(p)/p}{1 - 1/p}. \end{aligned}$$

Example. $f(T) = a + mT$ with $(a, m) = 1$. Then $\omega_f(p) = 0$ if $p \mid m$ and $\omega_f(p) = 1$ otherwise, so $C_f = \prod_{p \mid m} \frac{1}{1 - 1/p} = \frac{m}{\varphi(m)}$. ✓

Trying out the new heuristic for primes of the form n^2+1,

$$\pi_{T^2+1}(x) \overset{?}{\sim} \sum_{n\le x} \frac{C}{\log(n^2+1)} \sim \sum_{n\le x} \frac{C}{2\log n} \sim \frac{C}{2}\frac{x}{\log x},$$

where

$$C = \prod_{p} \frac{1-\omega_{T^2+1}(p)/p}{1-1/p} = \prod_{p>2}\left(1 - \frac{(-1|p)}{p-1}\right) \approx 1.37281346.$$

In table below, "Approx." comes from the first summation up to x.

x	10^4	10^5	10^6	10^7	10^8
$\pi_{T^2+1}(x)$	841	6656	54110	456362	3954181
Approx.	857	6612	53972	456406	3955221
Ratio	.9813	1.0066	1.0025	.9999	.9997

Compared to the bad guess $\pi_{T^2+1}(x) \sim x/2\log x$ before, this new guess looks good! (We also see what the old 1.3724 should be.)

For numerical testing we need a highly accurate approximation

$$C = \prod_{p>2} \left(1 - \frac{(-1|p)}{p-1}\right) \approx 1.37281346,$$

but the product converges *too slowly* to get many digits in C.

n	10^2	10^3	10^4	10^5
Product for $p \leq n$	1.3515	1.3704	1.3710	1.3723

Can speed up convergence of the infinite product by rewriting it:

$$C = \frac{4}{\pi} \prod_{p \equiv 1 \bmod 4} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p \equiv 3 \bmod 4} \left(1 - \frac{1}{p^2 - 1}\right).$$

n	10^2	10^3	10^4	10^5
Product for $p \leq n$	1.3727385	1.3728144	1.3728133	1.3728134

To make Hypothesis H for multiple polynomials $f_1(T), \ldots, f_r(T)$ quantitative, our basic heuristic is that for each prime p,

$$\mathrm{Prob}(n_1, \ldots, n_r \not\equiv 0 \bmod p) = \left(1 - \frac{1}{p}\right)^r$$

because the events $n_i \not\equiv 0 \bmod p$ are "independent," while

$$\mathrm{Prob}(f_1(n), \ldots, f_r(n) \not\equiv 0 \bmod p) = 1 - \frac{\omega_f(p)}{p}$$

where $f = f_1 \cdots f_r$.

Conjecture (Bateman–Horn, 1962) *For distinct nonconstant irreducible $f_1, \ldots, f_r$ in $\mathbf{Z}[T]$ with positive leading coefficients and a product f that satisfies the Bunyakovsky condition,*

$$\pi_{f_1,\ldots,f_r}(x) \overset{?}{\sim} \prod_p \left(\frac{1 - \omega_f(p)/p}{(1 - 1/p)^r}\right) \cdot \sum_{n \leq x} \frac{1}{\log f_1(n) \cdots \log f_r(n)}$$

$$\sim \frac{1}{\deg f_1 \cdots \deg f_r} \prod_p \left(\frac{1 - \omega_f(p)/p}{(1 - 1/p)^r}\right) \cdot \frac{x}{(\log x)^r}.$$

Twin primes

$$\pi_{T,T+2}(x) \overset{?}{\sim} \sum_{2\le n\le x} \frac{C}{(\log n)(\log(n+2))} \sim C\frac{x}{(\log x)^2},$$

where

$$C = \prod_p \frac{1-\omega_{T(T+2)}(p)/p}{(1-1/p)^2} = 2\prod_{p>2}\frac{1-2/p}{(1-1/p)^2} \approx 1.32032363.$$

In table below, "Approx." comes from the summation up to x.

x	10^4	10^5	10^6	10^7	10^8
$\pi_{T,T+2}(x)$	205	1224	8169	58980	440312
Approx.	213	1248	8247	58753	440367
Ratio	.9599	.9807	.9904	1.0038	.9998

This estimate for twin prime growth, as well as the good one for $\pi_{T^2+1}(x)$, goes back to Hardy and Littlewood (1923). They did not think in terms of probability, writing "*Probability* is not a notion of pure mathematics, but of philosophy or physics."

Twin primes vs. other prime pairs

Let's compare the growth of prime pairs $p, p+2$ (twin primes) to

- $p, p+4$ (cousin primes)
- $p, p+6$ (sexy primes)

Each should grow like $Cx/(\log x)^2$ for a $C > 0$. For twin primes

$$C_{T,T+2} = \frac{1-1/2}{(1-1/2)^2} \prod_{p>2} \frac{1-2/p}{(1-1/p)^2} = 2 \prod_{p>2} \frac{1-2/p}{(1-1/p)^2}$$

while for $p, p+4$

$$C_{T,T+4} = \frac{1-1/2}{(1-1/2)^2} \prod_{p>2} \frac{1-2/p}{(1-1/p)^2} = C_{T,T+2}$$

and

$$C_{T,T+6} = \frac{1-1/2}{(1-1/2)^2} \frac{1-1/3}{(1-1/3)^2} \prod_{p>3} \frac{1-2/p}{(1-1/p)^2} = \frac{1-1/3}{1-2/3} C_{T,T+2}.$$

Prime pairs $p, p+4$ and $p, p+6$

From $C_{T,T+4} = C_{T,T+2}$, we expect $\pi_{T,T+4}(x) \sim \pi_{T,T+2}(x)$.

$C_{T,T+6} = \frac{2/3}{1/3} C_{T,T+2} = 2C_{T,T+2}$: $\pi_{T,T+6}(x) \overset{?}{\sim} 2\pi_{T,T+2}(x)$.

The conjectural estimates

$$\pi_{T,T+4}(x) \sim \pi_{T,T+2}(x) \text{ and } \pi_{T,T+6}(x) \sim 2\pi_{T,T+2}(x)$$

don't require any approximation formulas to test, but only direct counts of prime values. See the table below.

x	10^4	10^5	10^6	10^7	10^8
$\pi_{T,T+2}(x)$	205	1224	8169	58980	440312
$\pi_{T,T+4}(x)$	203	1216	8144	58622	440258
$\pi_{T,T+6}(x)$	411	2447	16386	117207	879908
$2\pi_{T,T+2}(x)$	410	2448	16338	117960	880624

Prime pairs from n and $n+1$

The numbers n and $n+1$ are not both prime aside from 2 and 3, and $C_{T,T+1} = 0$. The only problem is at the prime 2. Set

$$n_{\text{odd}} = \text{ odd part of } n.$$

n	1	2	3	4	5	6	7	8	9	10
n_{odd}	1	1	3	1	5	3	7	1	9	5
$(n+1)_{\text{odd}}$	1	3	1	5	3	7	1	9	5	11

Can n_{odd} and $(n+1)_{\text{odd}}$ both be prime for infinitely many n?

x	10^4	10^5	10^6	10^7	10^8
Both prime for $n \leq x$	505	2933	18548	130005	962073

Exercise: Formulate a heuristic probability

$$\text{Prob}(n_{\text{odd}} \text{ is prime}) = \frac{C}{\log n_{\text{odd}}}$$

for a suitable $C > 0$ and n not a power of 2 (so $n_{\text{odd}} \neq 1$), and use it to conjecture how often n_{odd} and $(n+1)_{\text{odd}}$ are both prime.