1. a) Prove $U(\mathbf{Z}_p[u]_{u^2})$ is cyclic for all p.

   b) Prove $U(\mathbf{Z}_2[u]_{u^3})$ is cyclic and $U(\mathbf{Z}_2[u]_{u^k})$ is not cyclic for $k \geq 4$.

   c) Prove $U(\mathbf{Z}_p[u]_{u^k})$ is not cyclic if $p > 2$ and $k \geq 3$.

   d) For irreducible π in $\mathbf{Z}_p[u]$ with degree at least 2, prove $U(\mathbf{Z}_p[u]_{\pi^k})$ is not cyclic if $k \geq 2$.

2. Find all monic irreducibles of degree 4 in $\mathbf{Z}_3[u]$. (The number of them is 18.) If $f(T) = T^2 + u$ in $\mathbf{Z}_3[u][T]$, show $\pi_f(2) = 4$, where $\pi_f(n) = |\{g \in \mathbf{Z}_3[u] : \deg g = n, f(g) \text{ irreducible}\}|$.

3. For each irreducible π in $\mathbf{Z}_p[u]$ show $T^{\mathrm{N}(\pi)} - T + \pi$ (where $\mathrm{N}(\pi) = p^{\deg \pi}$) fails the Bunyakovsky condition at π. (This polynomial is irreducible in $\mathbf{Z}_p[u][T]$.)

4. For $f(T) \in \mathbf{Z}_p[u][T]$ with $\deg_T f > 0$, prove the following conditions are equivalent.

   (i) For each irreducible π in $\mathbf{Z}_p[u]$ there is $g \in \mathbf{Z}_p[u]$ such that $f(g) \not\equiv 0 \bmod \pi$.

   (ii) There are g and h in $\mathbf{Z}_p[u]$ such that $(f(g), f(h)) = 1$ in $\mathbf{Z}_p[u]$.

5. For $f(T) \in \mathbf{Z}_p[u][T]$ with $\deg_T f > 0$ we set $C_f = \prod_\pi (1 - \omega_f(\pi)/\mathrm{N}(\pi))/(1 - 1/\mathrm{N}(\pi))$, where $\omega_f(\pi) = |\{g \bmod \pi : f(g) \equiv 0 \bmod \pi\}|$.

   a) If $f(T) = T^p + u$ in $\mathbf{Z}_p[u][T]$, then prove $C_f = 1$.

   b) If $f(T) = F(T^p)$ in $\mathbf{Z}_p[u][T]$, then prove $C_f = C_F$. (The case $f(T) = T^p + u$ has $F(T) = T + u$.)

6. For prime $p > 2$, use results from the special number theory problem set on discriminants and resultants to prove the following.

   a) For nonconstant g in $\mathbf{Z}_p[u]$ with leading term cu^n, $\mu(g^p + u) = (-1)^n (\frac{c}{p})^n (\frac{-1}{p})^{n(pn-1)/2}$. In particular, $\mu(g^p + u) = 1$ if $4 \mid \deg g$.

   b) For all $g \in \mathbf{Z}_p[u]$, $\mu(g^{4p} + u^{2p-1})$ is 0 if $g(0) = 0$ and is 1 if $g(0) \neq 0$. In particular, $g^{4p} + u^{2p-1}$ is not irreducible for all $g \in \mathbf{Z}_p[u]$.