1. Use the Prime Number Theorem in the form $p_n \sim n \log n$ to prove for each $x > 0$ that $p_{[nx]}/p_n \to x$ as $n \to \infty$. This shows explicitly that prime ratios are dense in $(0, \infty)$.

2. For $f(T) \in \mathbf{Z}[T]$ with a positive leading coefficient, let $\pi_f(x) = |\{n \leq x : f(n) \text{ is prime}\}|$ and $p_{f,n}$ be the nth prime in the set $f(\mathbf{Z}^+)$. For any $C > 0$, show $\pi_f(x) \sim Cx/\log x$ as $x \to \infty$ if and only if $p_{f,n} \sim (1/C) n \log n$ as $n \to \infty$.

3. a) If $f_1(T) = T + c_1$ and $f_2(T) = T + c_2$ with $c_1 \neq c_2$ then prove the constant in the Bateman–Horn conjecture for how often $f_1(n)$ and $f_2(n)$ are both prime is a rational multiple of the constant in the quantitative form of the twin prime conjecture.

   b) Prove the conclusion of part a remains true if $f_1(T)$ and $f_2(T)$ are any two linear polynomials fitting the Bateman–Horn conjecture together (so we don't need to assume in part a that the leading coefficients equal 1).

4. If $d \in \mathbf{Z}$ is not a perfect square, show the Bateman–Horn conjecture for $f(T) = T^2 - d$ is

$$|\{n \leq x : n^2 - d \text{ is prime}\}| \sim \frac{C}{2} \frac{x}{\log x}$$

   where

$$C = \prod_{p \nmid 2d} \left(1 - \left(\frac{d}{p}\right) \frac{1}{p-1}\right).$$

5. If you know calculus, show for each $r \in \mathbf{Z}^+$ that $\displaystyle\sum_{2 \leq n \leq x} \frac{1}{(\log n)^r} \sim \frac{x}{(\log x)^r}$ as $x \to \infty$.

6. For an integer-valued polynomial $f(T)$, the count $\omega_f(p) = |\{n \bmod p : f(n) \equiv 0 \bmod p\}|$ might not make sense since $f(T)$ might not be a well-defined function on $\mathbf{Z}_p$. However, the ratio $\omega_f(p)/p$ can be generalized to have a meaning for integer-valued polynomials.

   a) If p is prime and $r \geq 1$ is chosen so that $p^r > \deg f$ show the ratio

$$\delta_f(p) = \frac{|\{n \bmod p^r : f(n) \equiv 0 \bmod p\}|}{p^r}$$

   is the same for all such r (the second modulus in the numerator is p, not p^r), and when $f(T) \in \mathbf{Z}[T]$ show this ratio is $\omega_f(p)/p$.

   b) If $f(T) = (T^2 + T + 2)/2$, compute $\delta_f(2)$, and if $f(T) = (T^3 + 2T - 6)/3$, compute $\delta_f(2)$ and $\delta_f(3)$.

   Remark. The Bateman–Horn conjecture for integer-valued polynomials $f_1(T), \ldots, f_r(T)$ that are irreducible in $\mathbf{Q}[T]$ with positive leading coefficients says for $f = f_1 \cdots f_r$ that

$$|\{n \leq x : f_1(n), \ldots, f_r(n) \text{ are all prime}\}| \sim \frac{1}{(\deg f_1) \cdots (\deg f_r)} \prod_p \frac{1 - \delta_f(p)}{(1 - 1/p)^r} \frac{x}{(\log x)^r}.$$