1. Prove there are infinitely many primes $p \equiv 1 \bmod 3$ and infinitely many primes $p \equiv 2 \bmod 3$. (Hint for $p \equiv 1 \bmod 3$: if $-3 \equiv \square \bmod p$ and $p \neq 2$ or 3 then $p \equiv 1 \bmod 3$.)

2. A nonzero $f(T) \in \mathbf{Z}[T]$ is called *reducible* in $\mathbf{Z}[T]$ if $f(T) = g(T)h(T)$ where $g(T)$ and $h(T)$ are not ± 1. A nonzero polynomial in $\mathbf{Z}[T]$ that is not reducible or ± 1 is called *irreducible*.

   Prove $f(T) \in \mathbf{Z}[T]$ is irreducible in $\mathbf{Z}[T]$ if and only if $f(T)$ is irreducible in $\mathbf{Q}[T]$ and primitive.

3. Let $f(T) \in \mathbf{Z}[T]$ be nonconstant. The *Bunyakovsky condition* on $f(T)$ is that for each prime p there is an $n \in \mathbf{Z}$ such that $p \nmid f(n)$. This is equivalent to saying $\gcd(f(n) : n \in \mathbf{Z}) = 1$.

   a) Determine which of the following four irreducibles in $\mathbf{Z}[T]$ satisfy the Bunyakovsky condition: $2T^3 + T^2 - 9T - 4$, $T^3 + T^2 - 2T - 3$, $2T^4 + 3T^3 + T + 5$, and $T^4 - T^2 + 6$. For the ones satisfying the condition, determine their first four prime values on $\mathbf{Z}^+$.

   b) For all $k \geq 0$ show $T^{2^k} + 1$ fits the Bunyakovsky condition. (These are irreducible in $\mathbf{Z}[T]$.)

   c) Find the first four $n \geq 1$ making $n^4 + 1$ prime and the first four $n \geq 1$ making $n^8 + 1$ prime.

   d) Prove $f(T)$ satisfies the Bunyakovsky condition if and only if there are distinct m and n in $\mathbf{Z}$ such that $(f(m), f(n)) = 1$. (Hint for "only if" direction: use any m for which $f(m) \neq 0$ and use the Chinese remainder theorem to select a corresponding n.)

4. Check the set of four polynomials $\{T, T+2, T+6, T+8\}$ fits the conditions of Hypothesis H, while the set of five polynomials $\{T, T+2, T+6, T+8, T+14\}$ does not.

5. A polynomial $f(T)$ is called *integer-valued* if $f(n) \in \mathbf{Z}$ for all $n \in \mathbf{Z}$. Any polynomial in $\mathbf{Z}[T]$ is integer-valued, but so are $(T^2 + T)/2$ and $(T^3 - T)/3$.

   a) For $k \geq 0$, the kth *binomial coefficient polynomial* is $\binom{T}{k} = \frac{T(T-1)\cdots(T-(k-1))}{k!}$. Check $\binom{-T}{k} = (-1)^k \binom{T+k-1}{k}$ and conclude that $\binom{T}{k}$ is integer-valued.

   b) Show integer-valued polynomials are exactly the $\mathbf{Z}$-linear combinations of binomial coefficient polynomials: there are $c_i \in \mathbf{Z}$ such that $f(T) = c_d\binom{T}{d} + \cdots + c_1\binom{T}{1} + c_0$.

   c) If $f(T)$ is integer-valued, prove $\gcd(f(n) : n \in \mathbf{Z}) = \gcd(c_0, \ldots, c_d)$.

   d) Write each of the four polynomials in Exercise 3a as a $\mathbf{Z}$-linear combination of binomial coefficient polynomials, and use 5c to determine which satisfy the Bunyakovsky condition.

   e) Hypothesis H for a nonconstant integer-valued polynomial $f(T)$ says that if $f(T)$ has a positive leading coefficient and is irreducible in $\mathbf{Q}[T]$, then $f(n)/g$ is prime for infinitely many $n \geq 1$, where $g = \gcd(f(n) : n \in \mathbf{Z})$. (Bunyakovsky's condition is the case $g = 1$.)

   The polynomials $T^2 + T + 2$ and $T^3 + 2T - 6$ are both irreducible in $\mathbf{Q}[T]$. Check the first polynomial has $g = 2$ and the second has $g = 3$, and find four $n \geq 1$ making $(n^2 + n + 2)/2$ prime and four $n \geq 1$ making $(n^3 + 2n - 6)/3$ prime.