Infinite Descent

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Fermat's original idea

> *As ordinary methods, such as are found in the books, are inadequate to proving such difficult propositions, I discovered at last a most singular method ... which I called the infinite descent.* Fermat, 1659

The idea: to prove an equation has no integral solutions, show one solution forces the existence of a smaller solution, leading to

$$a_1 > a_2 > a_3 > \cdots > 0,$$

which is impossible in $\mathbf{Z}^+$.
Ordinary mathematical induction could be considered infinite ascent, from n to $n+1$.

Outline

- Irrationality
- Nonsolvability of several equations in **Z** and **Q**
- Sums of Two Squares

Irrationality of $\sqrt{2}$

Here is the usual proof.
Suppose

$$\sqrt{2} = \frac{m}{n},$$

with m and n in $\mathbf{Z}^+$. Without loss of generality, $(m, n) = 1$. Then

$$m^2 = 2n^2,$$

so m^2 is even, so m is even: $m = 2m'$. Substitute and cancel:

$$2m'^2 = n^2.$$

Thus n^2 is even, so n is even. This contradicts $(m, n) = 1$.

Irrationality of $\sqrt{2}$

Here is a proof by descent. We don't have to insist $(m, n) = 1$. Suppose

$$\sqrt{2} = \frac{m}{n},$$

with m and n in $\mathbf{Z}^+$. Then

$$m^2 = 2n^2,$$

so m^2 is even, so m is even: $m = 2m'$. Substitute and cancel:

$$2m'^2 = n^2.$$

Thus n^2 is even, so n is even: $n = 2n'$, so

$$m'^2 = 2n'^2.$$

A solution (m, n) to $x^2 = 2y^2$ in $\mathbf{Z}^+$ leads to another (m', n') where $0 < m' < m$ (or $0 < n' < n$): a contradiction.

Irrationality of $\sqrt{2}$

Here is a wholly different proof by descent.
Suppose $\sqrt{2} \in \mathbf{Q}$. Since $1 < \sqrt{2} < 2$,

$$\sqrt{2} = 1 + \frac{a}{b}, \quad \text{with } 0 < \frac{a}{b} < 1.$$

Square both sides and clear the denominator:

$$2b^2 = b^2 + 2ab + a^2.$$

Thus $a^2 = b^2 - 2ab = (b - 2a)b$, so

$$\frac{a}{b} = \frac{b - 2a}{a}.$$

Now

$$\sqrt{2} = 1 + \frac{a}{b} = 1 + \frac{b - 2a}{a},$$

with a smaller denominator: $0 < a < b$. By descent we have a contradiction. (Or the denominator is eventually 1: $\sqrt{2} \in \mathbf{Z}$.)

Irrationality of $\sqrt{d}$

Let $d \in \mathbf{Z}^+$ with $d \neq \square$.
Suppose $\sqrt{d} \in \mathbf{Q}$. Let $\ell < \sqrt{d} < \ell + 1$, $\ell \in \mathbf{Z}$. Write

$$\sqrt{d} = \ell + \frac{a}{b}, \quad \text{with } 0 < \frac{a}{b} < 1.$$

Square both sides and clear the denominator:

$$db^2 = \ell^2 b^2 + 2\ell ab + a^2.$$

Thus $a^2 = db^2 - \ell^2 b^2 - 2\ell ab = (db - \ell^2 b - 2\ell a)b$ so

$$\frac{a}{b} = \frac{db - \ell^2 b - 2\ell a}{a}.$$

Now

$$\sqrt{d} = \ell + \frac{a}{b} = \ell + \frac{db - \ell^2 b - 2\ell a}{a},$$

with a smaller denominator: $0 < a < b$. By descent we have a contradiction. (Or the denominator is eventually 1: $\sqrt{d} \in \mathbf{Z}$.)

Impossibility of $x^2 + y^2 = 3$ in Q

Theorem

There is no solution to $x^2 + y^2 = 3$ in rational numbers.

If there is, x and y are not 0. We can take them both positive. Write $x = a/c$ and $y = b/c$ with a, b, c in $\mathbf{Z}^+$, so

$$a^2 + b^2 = 3c^2.$$

Then $a^2 + b^2 \equiv 0 \bmod 3$, so (!) a and b are multiples of 3: $a = 3a'$ and $b = 3b'$. Then

$$9a'^2 + 9b'^2 = 3c^2 \Longrightarrow 3(a'^2 + b'^2) = c^2,$$

so $3|c$: $c = 3c'$. Then

$$3(a'^2 + b'^2) = 9c'^2 \Longrightarrow a'^2 + b'^2 = 3c'^2.$$

We have a new solution with $0 < c' < c$: contradiction.

$x^4 + y^4 = z^2$

Theorem (Fermat)

There is no solution in $\mathbf{Z}^+$ *to* $x^4 + y^4 = z^2$.

This is the *only* result for which we have details of his proof!

Corollary

The equation $a^4 + b^4 = c^4$ *has no solution in* $\mathbf{Z}^+$.

To prove the theorem, let's make the Pythagorean triple (x^2, y^2, z) primitive. If a prime p divides x and y then $z^2 = x^4 + y^4$ is divisible by p^4: $p^4 | z^2$, so $p^2 | z$.

$$x = px', y = py', z = p^2 z' \Rightarrow p^4(x'^4 + y'^4) = p^4 z'^2.$$

Thus $x'^4 + y'^4 = z'^2$. So without loss of generality, $(x, y) = 1$.

$x^4+y^4=z^2$

When $x^4+y^4=z^2$ in $\mathbf{Z}^+$ with $(x,y)=1$, (x^2,y^2,z) is a primitive triple: one of x or y is odd and the other even. By symmetry, take x odd and y even, so

$$x^2=u^2-v^2, \quad y^2=2uv, \quad z=u^2+v^2$$

where $u>v>0$ and $(u,v)=1$ (and $u \not\equiv v \bmod 2$). Then (x,v,u) is a primitive triple with x odd, so v is even:

$$x=s^2-t^2, \quad v=2st, \quad u=s^2+t^2,$$

where $s>t>0$ and $(s,t)=1$. Note $z>u^2\geq u=s^2+t^2$, and

$$y^2=2uv=2(s^2+t^2)(2st)=4st(s^2+t^2).$$

$x^4 + y^4 = z^2$

$$y^2 = 4st(s^2 + t^2), \quad (s, t) = 1, \quad z > s^2 + t^2.$$

Since y is even,

$$\left(\frac{y}{2}\right)^2 = st(s^2 + t^2).$$

The factors on the right are pairwise relatively prime (why?) and each is positive, so they are all squares:

$$s = x'^2, \quad t = y'^2, \quad s^2 + t^2 = z'^2.$$

where x', y', z' are positive and pairwise relatively prime. Then

$$x'^4 + y'^4 = z'^2,$$

so we have a second primitive solution to our equation. Since

$$z > s^2 + t^2 = z'^2 \geq z',$$

we are done by descent on z: $z' < z$. Put differently, if $x^4 + y^4 = z^2$ has soln in $\mathbf{Z}^+$, so does $x^4 + y^4 = 1$, but it doesn't.

Summary of the descent

$$x^4 + y^4 = z^2, \quad (x, y) = 1, \quad y \text{ even},$$

$$x^2 = u^2 - v^2, \quad y^2 = 2uv, \quad z = u^2 + v^2, \quad (u, v) = 1,$$

$$x = s^2 - t^2, \quad v = 2st, \quad u = s^2 + t^2, \quad (s, t) = 1,$$

$$s = x'^2, \quad t = y'^2, \quad s^2 + t^2 = z'^2 \Rightarrow x'^4 + y'^4 = z'^2.$$

Suppose we started with $x^4 + y^4 = z^4$. Then what happens?

$$x^4 + y^4 = z^4, \quad (x, y) = 1, \quad y \text{ even},$$

$$x^2 = u^2 - v^2, \quad y^2 = 2uv, \quad z^2 = u^2 + v^2, \quad (u, v) = 1,$$

$$x = s^2 - t^2, \quad v = 2st, \quad u = s^2 + t^2, \quad (s, t) = 1,$$

$$s = x'^2, \quad t = y'^2, \quad s^2 + t^2 = z'^2 \Rightarrow x'^4 + y'^4 = z'^2.$$

Alternate Descent Parameter

The first solution (x, y, z) to $x^4 + y^4 = z^2$ can be written in terms of the second (smaller) solution (x', y', z'):

$$x = x'^4 - y'^4, \quad y = 2x'y'z', \quad z = 4x'^4y'^4 + z'^4.$$

So in fact $z > z'^4$, not just $z > z'^2$ as before. These explicit formulas tell us

$$0 < y' < y \text{ and } 0 < \max(x', y') < y \leq \max(x, y),$$

so we could do descent on $\max(x, y)$ (on y?) rather than on z.

Consequences of nonsolvability of $x^4+y^4=z^2$ in $\mathbf{Z}^+$

Corollary

Any integral solution to $x^4+y^4=z^2$ has x or y equal to 0.

Otherwise change signs to make x and y (and z) all positive.

Corollary

The only rational solutions to $y^2=x^4+1$ are $(0,\pm 1)$.

Set $x=a/c$ and $y=b/c$ to get $(bc)^2=a^4+c^4$. Thus $a=0$, so $x=0$.

Corollary

The only rational solutions to $2y^2=x^4-1$ are $(\pm 1,0)$.

Square and fiddle to get $(y/x)^4+1=((x^4+1)/2x^2)^2$, so $y=0$.

Consequences of nonsolvability of $x^4 + y^4 = z^2$ in $\mathbf{Z}^+$

Corollary

The only rational solutions to $y^2 = x^3 - 4x$ *are* $(0,0), (\pm 2, 0)$.

There is a one-to-one correspondence

$$v^2 = u^4 + 1 \longleftrightarrow y^2 = x^3 - 4x, \; x \neq 0.$$

given by

$$x = \frac{2}{u^2 - v} \qquad y = \frac{4u}{u^2 - v}$$
$$u = \frac{y}{2x} \qquad v = \frac{y^2 - 8x}{4x^2},$$

so from the corollary that $v^2 = u^4 + 1$ only has rational solutions with $u = 0$, rational solutions to $y^2 = x^3 - 4x$ have $x = 0$ or $y = 0$.

Consequences of nonsolvability of $x^4 + y^4 = z^2$ in $\mathbf{Z}^+$

Corollary

The only rational solution to $y^2 = x^3 + x$ is $(0, 0)$.

Assume $x \neq 0$. Since $y^2 = x(x^2+1)$, $y \neq 0$. May take $x, y > 0$. Then (!) $x = a/c^2$ and $y = b/c^3$ in reduced form, so

$$\left(\frac{b}{c^3}\right)^2 = \left(\frac{a}{c^2}\right)^3 + \frac{a}{c^2} \Longrightarrow b^2 = a^3 + ac^4 = a(a^2 + c^4).$$

Since $(a, c) = 1$,

$$a = u^2, \quad a^2 + c^4 = v^2 \Longrightarrow u^4 + c^4 = v^2.$$

$x^4 - y^4 = z^2$

Theorem (Fermat)

There is no solution in $\mathbf{Z}^+$ *to* $x^4 - y^4 = z^2$.

To prove the theorem, since $z^2 + y^4 = x^4$ instead of $x^4 + y^4 = z^2$, reverse the roles of x and z; do descent on x instead of on z. Some extra details arise. On the right side below are explicit formulas for a solution (x, y, z) in terms of a "smaller" solution (x', y', z').

$x^4 + y^4 = z^2$	$x^4 - y^4 = z^2$
$x = x'^4 - y'^4$	$x = x'^4 + y'^4$
$y = 2x'y'z$	$y = 2x'y'z'$
$z = 4x'^4y'^4 + z'^4$	$z = \lvert 4x'^4y'^4 - z'^4 \rvert$
$z' \leq z'^4 < z$	$x' \leq x'^4 < x$

Consequences of nonsolvability of $x^4 - y^4 = z^2$ in $\mathbf{Z}^+$

Old corollaries	New corollaries
$x^4 + y^4 = z^2$ in $\mathbf{Z} \Rightarrow xy = 0$	$x^4 - y^4 = z^2$ in $\mathbf{Z} \Rightarrow yz = 0$
$y^2 = x^4 + 1$ in $\mathbf{Q} \Rightarrow x = 0$	$y^2 = x^4 - 1$ in $\mathbf{Q} \Rightarrow y = 0$
$2y^2 = x^4 - 1$ in $\mathbf{Q} \Rightarrow x = \pm 1$	$2y^2 = x^4 + 1$ in $\mathbf{Q} \Rightarrow x = \pm 1$
$y^2 = x^3 - 4x$ in $\mathbf{Q} \Rightarrow y = 0$	$y^2 = x^3 + 4x$ in $\mathbf{Q} \Rightarrow y = 0$
$y^2 = x^3 + x$ in $\mathbf{Q} \Rightarrow y = 0$	$y^2 = x^3 - x$ in $\mathbf{Q} \Rightarrow y = 0$

Consequences of nonsolvability of $x^4 \pm y^4 = z^2$ in $\mathbf{Z}^+$

Theorem

No Pythagorean triple has two terms that are squares.

Otherwise we could solve $x^4 + y^4 = z^2$ or $x^4 + y^2 = z^4$ in $\mathbf{Z}^+$.
Many Pythagorean triples have one term that is a square:

a	3	7	9	16	17	225
b	4	24	40	63	144	272
c	5	25	41	65	145	353

Theorem

The only triangular number that is a fourth power is 1*.*

If $m(m+1)/2 = n^4$ with $m > 1$ then $\{m, m+1\} = \{x^4, 2y^4\}$ with $x > 1$ and $y > 1$, so $x^4 - 2y^4 = \pm 1 \Longrightarrow y^8 \pm x^4 = ((x^4 \pm 1)/2)^2$.
This is impossible in positive integers.

Consequences of nonsolvability of $x^4 \pm y^4 = z^2$ in $\mathbf{Z}^+$

Why did Fermat look at $x^4 \pm y^4 = z^2$ rather than $x^4 \pm y^4 = z^4$?

Theorem (Fermat)

No Pythagorean triangle has area equal to a square or twice a square.

This first part was stated by Fibonacci (1225), without proof.

$a^2+b^2=c^2$, $\frac{1}{2}ab=d^2$	$x^4-y^4=z^2$	$a^2+b^2=c^2$, $\frac{1}{2}ab=2d^2$	$x^4+y^4=z^2$
$x=c$	$a=z^2$	$x=b$	$a=x^2$
$y=2d$	$b=2x^2y^2$	$y=2d$	$b=y^2$
$z=\lvert a^2-b^2\rvert$	$c=x^4+y^4$	$z=bc$	$c=z$
	$d=xyz$		$d=xy/2$

These are not inverse correspondences, but that's okay.

$x^3+y^3=z^3$

Theorem (Euler, 1768)

There is no solution in $\mathbf{Z}^+$ *to* $x^3+y^3=z^3$.

Euler used descent and needed a lemma.

Lemma

If $a^2+3b^2 =$ *cube and* $(a,b)=1$ *then* $a=u^3-9uv^2$ *and* $b=3u^2v-3v^3$ *for some* $u,v\in\mathbf{Z}$.

This is analogous to a description of $a^2+b^2 =$ cube with $(a,b)=1$: $a=u^3-3uv^2$ and $b=3u^2v-v^3$. Euler proved the lemma with unique factorization in $\mathbf{Z}[\sqrt{-3}]$, but that is *false*:

$$4=2\cdot 2=(1+\sqrt{-3})(1-\sqrt{-3}).$$

Nevertheless, the lemma is true!

Selmer's example

Theorem (Selmer, 1951)

The only integral solution to $3x^3 + 4y^3 = 5z^3$ is $(0, 0, 0)$.

It can be shown $3x^3 + 4y^3 \equiv 5z^3 \bmod n$ has a solution $\not\equiv (0, 0, 0) \bmod n$ for all $n \geq 2$, so nonsolvability in $\mathbf{Z}$ can't be seen by congruence considerations.

We sketch a proof of the theorem using descent. From an integral solution $(x, y, z) \neq (0, 0, 0)$, none of the terms is 0 and we get

$$3x^3 + 4y^3 = 5z^3 \Longrightarrow (2y)^3 + 6x^3 = 10z^3,$$

so

$$a^3 + 6b^3 = 10c^3$$

for $a = 2y, b = x, c = z$. May take a, b, c pairwise relatively prime.

Selmer's example

$$a^3+6b^3=10c^3, \quad (a,b,c)=1$$

Using $\mathbf{Z}[\sqrt[3]{6}]=\{k+\ell\sqrt[3]{6}+m\sqrt[3]{36} : k,\ell,m\in\mathbf{Z}\}$, basically get

$$a+b\sqrt[3]{6}=(2-\sqrt[3]{6})(1-\sqrt[3]{6})\alpha^3$$

for some $\alpha\in\mathbf{Z}[\sqrt[3]{6}]$. Write $\alpha=k+\ell\sqrt[3]{6}+m\sqrt[3]{36}$ and equate coefficients of $\sqrt[3]{36}$ on both sides above:

$$\begin{aligned} 0 &= k^3+6\ell^3+36m^3+36k\ell m+2(3k\ell^2+3k^2m+18\ell m^2) \\ &\quad -3(3k^2\ell+18km^2+18\ell^2m). \end{aligned}$$

Reduce mod 3: $0\equiv k^3$, so $3|k$. Reduce mod 9: $0\equiv 6\ell^3$, so $3|\ell$. Reduce mod 27: $0\equiv 36m^3$, so $3|m$. Divide by 3^3 and repeat again. Thus $\alpha=0$, so $a=b=0$, so $x=b=0$, $y=a/2=0$, $z=0$.

Fermat speaks

If there is a right triangle with integral sides and with an area equal to the square of an integer, then there is a second triangle, smaller than the first, which has the same property [...] and so on ad infinitum. [...] From which one concludes that it is impossible that there should be [such] a right triangle.
It was a long time before I was able to apply my method to affirmative questions, because the way and manner of getting at them is much more difficult than that which I employ with negative theorems. So much so that, when I had to prove that every prime number of the form $4k + 1$ *is made up of two squares, I found myself in much torment. But at last a certain meditation many times repeated gave me the necessary light, and affirmative questions yielded to my method [...]* Fermat, 1659

Affirmative Questions

Some positive theorems Fermat (1659) suggested he could prove by descent:

- Two Square Theorem: Any prime $p \equiv 1 \bmod 4$ is a sum of two squares (Euler, 1747)
- Four Square Theorem: Every positive integer is a sum of four squares (Lagrange, 1770).
- For $d \neq \square$, $x^2 - dy^2 = 1$ has infinitely many integral solutions (Lagrange, 1768). The difficult step is existence of even one nontrivial solution ($y \neq 0$).

Sums of Two Squares

Theorem

For prime p, if $-1 \equiv \square \bmod p$ then $p = x^2 + y^2$ in $\mathbf{Z}$.

By hypothesis, $-1 \equiv a^2 \bmod p$. May take $|a| \leq p/2$. Write

$$a^2 + 1 = pd,$$

so

$$pd = a^2 + 1 \leq \left(\frac{p}{2}\right)^2 + 1 = \frac{p^2}{4} + 1 < \frac{p^2}{2}$$

and thus $d < p/2$. From any equation with side condition

$$pk = x^2 + y^2, \quad 0 < k < \frac{p}{2}$$

where $k > 1$, we will find such an equation with $0 < k' < k$. So eventually $k = 1$ and p is sum of two squares! How do we get k'?

Sums of Two Squares

We have

$$pk = x^2 + y^2, \quad 1 < k < \frac{p}{2}.$$

Set $x \equiv r \bmod k$, $y \equiv s \bmod k$, with $|r|, |s| \leq k/2$. At least one of r and s is not 0: otherwise, $k|x$ and $k|y$, so $k^2|pk$, and thus $k|p$. But $1 < k < p$. Since

$$r^2 + s^2 \equiv x^2 + y^2 \equiv 0 \bmod k,$$

we can set $r^2 + s^2 = kk'$ with $k' > 0$. Then

$$0 < kk' = r^2 + s^2 \leq \left(\frac{k}{2}\right)^2 + \left(\frac{k}{2}\right)^2 = \frac{k^2}{2},$$

which makes $0 < k' \leq k/2 < k$. We will show pk' is a sum of two squares.

Sums of Two Squares

$$pk = x^2 + y^2, \quad kk' = r^2 + s^2, \quad x \equiv r \bmod k, \quad y \equiv s \bmod k.$$

Multiplying,

$$(pk)(kk') = (x^2+y^2)(r^2+s^2) = (xs - yr)^2 + (xr + ys)^2,$$

and modulo k, $xs - yr \equiv xy - yx \equiv 0$, $xr + ys \equiv x^2 + y^2 \equiv 0$. Write $xs - yr = kx'$ and $xr + ys = ky'$. Then

$$pk^2k' = (kx')^2 + (ky')^2 = k^2(x'^2 + y'^2).$$

Divide by k^2: $pk' = x'^2 + y'^2$, and $0 < k' < k$ (so $0 < k' < p/2$). Repeat until $k = 1$.

Remark. Fermat's own proof by descent that p is a sum of two squares used counterexamples: from one, get a smaller one. Eventually reach 5, which is not a counterexample!

Sums of Two Squares

Theorem

If $n \in \mathbf{Z}^+$ is a sum of two squares in $\mathbf{Q}$ then it is a sum of two squares in $\mathbf{Z}$.

Example

No solution to $21 = x^2 + y^2$ in $\mathbf{Q}$ since none in $\mathbf{Z}$.

Suppose $n = r^2 + s^2$ with rational r and s. Write $r = a/c$ and $s = b/c$ with common denominator $c \geq 1$. If $c > 1$, find a second representation $n = r'^2 + s'^2$ in $\mathbf{Q}$ with common denominator $0 < c' < c$. So eventually $c = 1$ and $n = a^2 + b^2$ in $\mathbf{Z}$.
The idea for this descent is geometric: get new pairs (r, s), (r', s'), (r'', s''), ... using repeated intersections of lines with the circle $x^2 + y^2 = n$ in $\mathbf{R}^2$.

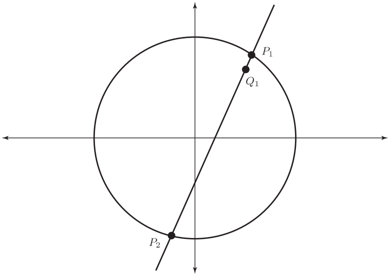
An Example

Start with $193 = (933/101)^2 + (1048/101)^2$. Let

$$P_1 = \left(\frac{933}{101}, \frac{1048}{101}\right) \approx (9.2, 10.3).$$

Its nearest integral point is $Q_1 = (9, 10)$, and the line $\overline{P_1 Q_1}$ meets the circle $x^2 + y^2 = 193$ in P_1 and

$$P_2 = \left(-\frac{27}{5}, -\frac{64}{5}\right).$$



An Example, contd.

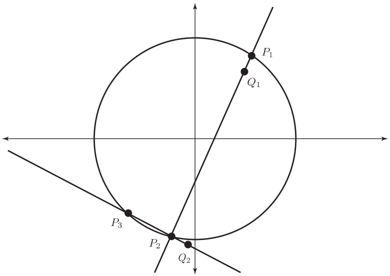
The nearest integral point to

$$P_2 = \left(-\frac{27}{5}, -\frac{64}{5}\right) = (-5.4, -12.8)$$

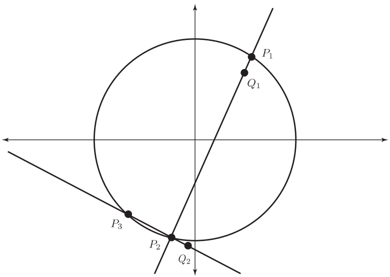
is $Q_2 = (-5, -13)$, and the line $\overline{P_2 Q_2}$ meets the circle in P_2 and the point

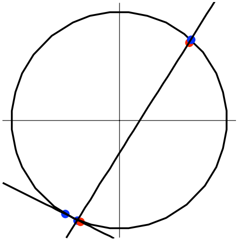
$$P_3 = (-7, -12).$$

$$193 = (-7)^2 + (-12)^2 = 7^2 + 12^2$$



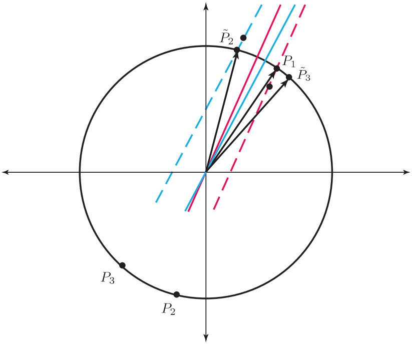
The Real Picture





Using Reflections

The second intersection point of a line with a circle could be replaced with reflection across a parallel line through the origin.



$$\widetilde{P}_2 = \left(\frac{27}{5}, \frac{64}{5}\right), \quad \widetilde{P}_3 = (7, 12)$$

Sums of Two Squares

Intersections of lines with a sphere in $\mathbf{R}^3$ works for three squares:

Theorem

If $n \in \mathbf{Z}^+$ is a sum of three squares in $\mathbf{Q}$ then it is a sum of three squares in $\mathbf{Z}$.

Start with $13 = (18/11)^2 + (15/11)^2 + (32/11)^2$.

$$P_1 = \left(\frac{18}{11}, \frac{15}{11}, \frac{32}{11}\right) \rightsquigarrow Q_1 = (2, 1, 3),$$

$\overline{P_1 Q_1}$ meets $x^2 + y^2 + z^2 = 13$ in P_1 and $P_2 = (2/3, 7/3, 8/3)$.

$$P_2 = \left(\frac{2}{3}, \frac{7}{3}, \frac{8}{3}\right) \rightsquigarrow Q_2 = (1, 2, 3),$$

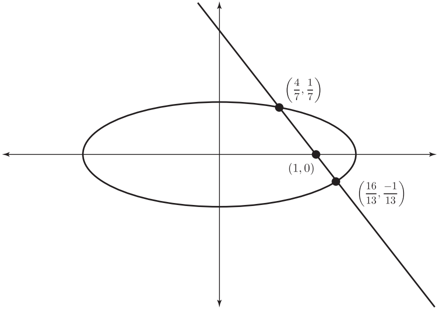
$\overline{P_2 Q_2}$ meets the sphere in P_2 and $P_3 = (0, 3, 2)$: $13 = 0^2 + 3^2 + 2^2$.

Cautionary examples

The equation

$$x^2 + 82y^2 = 2$$

has no integral solution, but it has the rational solution $(4/7, 1/7)$. What happens if we try the method of proof? The nearest integral point is $(1, 0)$ and the line through them meets the ellipse in $(16/13, -1/13)$: the denominator has gone up, not down.



Cautionary examples

The equation

$$x^3 + y^3 = 13$$

has no integral solution, but it has the rational solution $(7/3, 2/3)$. Its nearest integral point is $(2, 1)$, and the line through them meets the curve in $(2/3, 7/3)$, whose nearest integral point is $(1, 2), \ldots$.