QUATERNION ALGEBRAS: SET 3

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Unless stated otherwise, F is a field with characteristic not 2.

1. We know that $(2,3)_{\mathbf{Q}}$ and $(3,11)_{\mathbf{Q}}$ are division rings.

a) Prove they are isomorphic. (Can you find a specific $q \in (2,3)^0_{\mathbf{Q}}$ which squares to 11?)

b) Prove that for any non-zero integers a, b such that $(2,3)_{\mathbf{Q}} \cong (a,b)_{\mathbf{Q}}$, at least one of a and b is a multiple of 3.

c) Find a quaternion division algebra $(3,*)_{\mathbf{Q}}$ which is not isomorphic to $(2,3)_{\mathbf{Q}}$.

2. Show $\mathbf{H}(\mathbf{Q}) \cong (-5,-29)_{\mathbf{Q}}$. Is $\mathbf{H}(\mathbf{Q}) \cong (-2,-5)_{\mathbf{Q}}$?

3. Let $a, b, b' \in F^\times$.

a) If $(a,b)_F \cong \mathrm{M}_2(F)$, prove $(a,b')_F \cong (a,bb')_F$. (For example, if $p = 2$ or $p \equiv 1 \bmod 4$, then we already know $(p,-1)_{\mathbf{Q}} \cong \mathrm{M}_2(\mathbf{Q})$, so $(p,r)_{\mathbf{Q}} \cong (p,-r)_{\mathbf{Q}}$ for any $r \in \mathbf{Q}^\times$.) Is the converse true?

b) By part a, $(2,3)_{\mathbf{Q}} \cong (2,-3)_{\mathbf{Q}}$. Show $(-2,3)_{\mathbf{Q}} \cong \mathrm{M}_2(\mathbf{Q})$ and $(-2,-3)_{\mathbf{Q}} \cong \mathbf{H}(\mathbf{Q})$.

4. (Quadratic Dickson's lemma in action)

a) The quadratic Dickson's lemma tells us that in $\mathbf{H}$, $j = qiq^{-1}$ and $-i = \widetilde{q}i\widetilde{q}^{-1}$ for some q and $\widetilde{q}$ in $\mathbf{H}^\times$. Make this explicit (*i.e.*, find a q and $\widetilde{q}$) by working through the proof of Dickson's lemma in these examples. In $\mathbf{H}(\mathbf{Q})$, $(i + 2j + 2k)/3$ is a root of $T^2 + 1$, so it also must be conjugate to i. Make this conjugation explicit.

b) In $\mathbf{H}(\mathbf{Q})$, $q = 1/2 + i/6 + j/6 + 5k/6$ has trace 1 and norm 1. So does $q' = (1 + i + j + k)/2$. That makes them both roots of $T^2 - T + 1$. Use the proof of the quadratic Dickson's lemma to exhibit a conjugation between them.

5. Show the quadratic Dickson's lemma is true in $\mathrm{M}_2(F)$, where F has any characteristic. That is, if $f(T) \in F[T]$ is a quadratic irreducible polynomial and $x, y \in \mathrm{M}_2(F)$ are roots, then $y = qxq^{-1}$ for some invertible q in $\mathrm{M}_2(F)$.

6. (A special case of the quadratic Dickson's lemma via linear algebra) Let D be a finite-dimensional division ring over its center F, with $\dim_F D > 1$. Suppose $x \in D - F$ is a root of the quadratic polynomial $T^2 + c_1T + c_0 \in F[T]$. Another root is $y = -x - c_1$. Prove x and $-x - c_1$ are conjugate by considering the function $L\colon D \to D$ given by $L(d) = dx - yd$. Show $L(d)$ commutes with x for every $d \in D$, so L can't be onto (D is non-commutative). Use linear algebra to explain why $L(d) = 0$ for some non-zero d.

7. Let B be a quaternion algebra over a field F and let $f\colon B \to B$ be an F-algebra homomorphism. Then $\overline{f(q)} = f(\overline{q})$ for every $q \in B$. (Hint: Reduce to the case of pure quaternions q and show $f(B^0) \subset B^0$.)

8. (Characterizing quaternionic conjugation) Let R be a ring, possibly noncommutative. An *involution* on R is a map $a \mapsto a^*$ which satisfies the properties

- $(a + b)^* = a^* + b^*$,
- $(ab)^* = b^*a^*$,
- $a^{**} = a$,

for all $a, b \in R$, where a^{**} means $(a^*)^*$. Examples of involutions are conjugation on quadratic fields, transposition on matrices, and quaternionic conjugation on $(a,b)_F$ (when $\operatorname{char} F \neq 2$). Prove the following properties of involutions.

a) $1^* = 1$.
b) $(aa^*)^* = aa^*$.
c) $a \in R^\times \iff a^* \in R^\times$, in which case $(a^*)^{-1} = (a^{-1})^*$.
d) When R is non-commutative, $a^* \neq a$ for some $a \in R$.
e) a is in the center of R if and only if a^* is in the center of R.
f) Let B be a quaternion algebra over F, where $\operatorname{char} F \neq 2$. Show quaternionic conjugation on B is the unique involution on B which fixes F pointwise and which satisfies $qq^* \in F$ for every $q \in B$.
g) Can you find a mapping $q \mapsto q^*$ on $\mathbf{H}$ which satisfies the first two properties of an involution but not the third (that is, $q^{**} \neq q$ for some q)?

9. On the previous set, the isomorphism $(a, b)_F \cong (a + b, -ab)_F$ was obtained by an explicit change of basis. As a different argument, show $(a, -ab)_F \cong (a + b, -ab)_F$ using the theorem from lecture which describes when two quaternion algebras with a "common slot" are isomorphic. (Then the isomorphism $(a, b)_F \cong (a, -ab)_F$ finishes the proof.)

10. (Normalizing quadratic fields). Let F be a field of any characteristic and let K be a quadratic extension field of the F: $K = F[r]$ where r is the root of an irreducible quadratic polynomial in $F[T]$. There can be many such representations of a quadratic extension, *e.g.*, if α is a real root of $T^2 - 6T + 7$ and β is a real root of $T^2 + 2T - 17$, then $\mathbf{Q}[\alpha] = \mathbf{Q}[\beta] = \mathbf{Q}[\sqrt{2}]$. In some sense, the last representation, corresponding to the polynomial $T^2 - 2$, is nicest. We want to obtain such nice representations for most quadratic fields.

a) If $\operatorname{char} F \neq 2$ and $K = F[r]$ is a quadratic extension field, show a quadratic polynomial in $F[T]$ with r as a root has non-zero discriminant and then show we can write $K = F[s]$, where s is the root of a quadratic with the "normal form" $T^2 - c$ for some $c \in F^\times$. (Hint: complete the square of the polynomial with r as a root.)

b) If $\operatorname{char} F = 2$ *and* the quadratic polynomial with r as a root has non-zero discriminant, show $K = F[s]$ where s is the root of a quadratic with the "normal form" $T^2 + T + c$ for some $c \in F$. Also, show it is impossible to write this field K in the form $F[s']$ where s' is the root of a quadratic with the form $T^2 - c$ for some $c \in F^\times$. (Hint: if it is possible, prove $\alpha^2 \in F$ for every $\alpha \in K$, and get a contradiction from that.)

c) (a pathological quadratic extension) Let $F = \mathbf{F}_2(x)$ and $f(T) = T^2 - x$. Show $f(T)$ is an irreducible in $F[T]$ with discriminant 0. Setting $K = F[r]$ where $f(r) = 0$, show $\alpha^2 \in F$ for every $\alpha \in K$.

11. For $a \in F^\times$, show $(a, a)_F \cong (-1, a)_F$. Also, if $a, b \in F^\times$ satisfy $a + b = c^2$ for some $c \in F$, show $(a, b) \cong \mathrm{M}_2(F)$.

12. Show -1 is not a sum of two squares in the field $\mathbf{Q}[\sqrt{-7}]$, so $\mathbf{H}(\mathbf{Q}[\sqrt{-7}])$ is a division ring. Is $\mathbf{H}(\mathbf{Q}[\sqrt{-2}])$ a division ring? What about $\mathbf{H}(\mathbf{Q}[\sqrt{-3}])$?

13. For $A \in \mathrm{M}_2(F)$, show A^2 is a scalar matrix if and only if A is a scalar matrix or A has trace 0. Then determine if the analogous statement is true in any quaternion algebra $(a, b)_F$. (When $F = \mathbf{R}$, you looked at this in exercise 16c on set 1.)

14. If you know quadratic reciprocity for $\mathbf{F}[t]$, where $\mathbf{F}$ is a finite field with odd characteristic, then prove for $f \in \mathbf{F}(t)^\times$ that

$$(f, g)_{\mathbf{F}(t)} \cong \mathrm{M}_2(\mathbf{F}(t)) \text{ for all } g \neq 0 \iff f \in \mathbf{F}(t)^{\times 2}.$$

This is an analogue of the last exercise on the previous set.