ZETA AND L-FUNCTIONS
TECHNICAL HANDOUT 2
JULY 6, 2000

For concreteness, in this handout it is understood that a sequence a_n of complex numbers is indexed by the positive integers in their natural order, although in practice we might meet sequences indexed by other (possibly unordered) sets, such as the primes in $\mathbf{Z}^+$ or sets of primes in $\mathbf{Z}[i]$.

We will say a convergent infinite series $\sum a_n$ can be *rearranged* if any permutation of the terms and any collecting of the terms into "subsums" produces convergent infinite series with the same sum as the original series. This corresponds to the ideas of commutativity and associativity for finite sums. We can similarly use this terminology for infinite products. (By the way, the usual term used here is "absolutely convergent" rather than "can be rearranged." I choose to use a nonstandard terminology in order to emphasize what the importance of the property is rather than how it is checked.)

The practical importance of this property is that it avoids any real need to specify a definite ordering of the terms in the sum or product, which is important because many zeta and L-functions are sums or products over sets which don't have a strict linear ordering like the positive integers.

Convergence Theorem 0: If $a_n \geq 0$ and $\sum_{n=1}^N a_n$ has an upper bound which does not depend on N, then $\sum_{n\geq 1} a_n$ converges and can be rearranged. Otherwise $\sum_{n\geq 1} a_n = \infty$, and any rearrangement (in the above sense) also sums to ∞.

If $a_n \geq 0$ and some rearrangement of the series $\sum_{n\geq 1} a_n$ converges, then so does the original series.

Convergence Theorem 1: Let $\sum a_n$ and $\sum b_n$ converge where $a_n, b_n \geq 0$. Then $\sum(a_n+b_n)$ converges, equals $\sum a_n + \sum b_n$, can be rearranged, and also $\sum a_n \cdot \sum b_n = \sum c_n$, where c_n is any ordering of the terms $a_i b_j$.

By induction, this extends to any finite sum or product of infinite series.

Convergence Theorem 2: Suppose $c_n \geq 0$, $c_n \neq 1$, and $\sum c_n$ converges. Then

$$\prod_{n\geq 1} \frac{1}{1-c_n}, \quad \prod_{n\geq 1} (1-c_n)$$

both converge to nonzero numbers and can be rearranged. Moreover, if all $c_n < 1$, then these products can be expanded into rearrangeable series in the naive way (using geometric series for each factor of the first product).

Convergence Theorem 3: When $a_n \in \mathbf{C}$ and $\sum |a_n|$ converges, then $\sum a_n$ converges, can be rearranged, and $|\sum a_n| \leq \sum |a_n|$.

For $a_n, b_n \in \mathbf{C}$, the conclusion of Convergence Theorem 1 is true if $\sum |a_n|$ and $\sum |b_n|$ converge.

For $c_n \in \mathbf{C}$, the conclusion of Convergence Theorem 2 is true if $\sum |c_n|$ converges and $|c_n| < 1$ for all n.

Convergence Theorem 4: Let $f_n(s)$ be continuous functions for $a < s < b$. Suppose $|f_n(s)| \leq M_n$ for $a < s < b$, where $\sum M_n$ converges. Then $f(s) = \sum f_n(s)$ is a convergent rearrangeable series for each s and $f(s)$ is continuous in s.

Addendum to Convergence Theorem 2: $\log \prod 1/(1 - c_n) = - \sum \log(1 - c_n)$.