ZETA AND L-FUNCTIONS TECHNICAL HANDOUT JULY 3, 2000

Notational Conventions

Elem. no. thy.	No. thy.
$\mathbf{Z}_m$	$\mathbf{Z}/m$
$\mathbf{U}_m$	$(\mathbf{Z}/m)^\times$
$\mathbf{Z}_p$	$\mathbf{Z}/p$ or $\mathbf{F}_p$
$\mathbf{Z}_p[T]_{f(T)}$	$(\mathbf{Z}/p)[T]/f(T)$ or $\mathbf{F}_p[T]/f(T)$

A useful estimate to "avoid" calculus: For $y > 0$ and $\alpha > 0$,

$$y^\alpha \geq 1 + \alpha \frac{y-1}{y}.$$

Equivalently, $(1+x)^\alpha \geq 1 + \alpha x/(1+x)$ for $x > -1$ and $\alpha > 0$.

Facts about (infinite) series and products.

(1) $\sum_{n\geq 0} r^n = 1/(1-r)$ for $|r| < 1$.

(2) If $a_n \geq 0$, then $\sum_{n\geq 1} a_n$ converges if and only if the partial sums $S_N = \sum_{n=1}^N a_n$ are bounded above, in which case any rearrangement of the terms in $\sum a_n$ yields the same sum. (This is commutativity of series with non-negative terms.)

(3) If $a_1 > a_2 > a_3 > \cdots > 0$ and $a_n \to 0$, then $\sum_{n\geq 1} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$ converges. The order of the terms in the sum *might* matter!

(4) If $\sum |a_n|$ converges, then so does $\sum a_n$ and the terms can be rearranged.

(5) If $\sum |a_n|$ converges, with $a_n \neq 1$ for all n, then $\prod \frac{1}{1-a_n}$ converges to a nonzero number and the terms in the product can be rearranged. If $|a_n| \leq b < 1$ for a common value of b, then the infinite product $\prod \frac{1}{1-a_n}$ can be expanded into a series by writing out each factor as a geometric series $\sum_{k\geq 0} a_n^k$ and multiplying together these series in the naive manner.

(6) Let $f_n(x)$ be functions for $x \in (a,b)$. Suppose $|f_n(x)| \leq M_n$ for all $x \in (a,b)$ and $\sum M_n$ converges. Then

$$f(x) = \sum f_n(x)$$

converges for each x and the terms in the series can be rearranged. Moreover, if all f_n are continuous, then f is continuous. (Warning to those who know calculus: if all f_n are differentiable, or even infinitely differentiable, the summation function f does *not* have to inherit either of these nicer analytic properties.)