# ZETA AND $L$-FUNCTIONS
# HOMEWORK 5
# JULY 27, 2000

*Due: Thursday, August 3 at the beginning of class*

*Problems:*

(1) (An application of Dirichlet's and Kornblum's theorem)

a) Let $m|n$ in $\mathbf{Z}$ and $M|N$ in $\mathbf{F}_p[T]$. (Here $m, n, M, N$ are all nonzero.) Use the Chinese Remainder Theorem to show the reduction maps $(\mathbf{Z}/n)^\times \to (\mathbf{Z}/m)^\times$ and $(\mathbf{F}_p[T]/N)^\times \to (\mathbf{F}_p[T]/M)^\times$ are surjective.

b) Let $m$ be a positive integer, $a$ be relatively prime to $m$, and $S$ be a finite set of primes. Use Dirichlet's theorem and part a) to prove that

$$\gcd(\{p - a : p \equiv a \bmod m, p \notin S\}) = m$$

unless $m$ is odd and $a$ is odd, in which case the gcd is $2m$. Try to give a proof that works when $S$ is a possibly infinite set of primes, with a suitable constraint on $S$.

c) Let $M$ be monic in $\mathbf{F}_p[T]$, $A$ be relatively prime to $M$, and $S$ be a finite set of monic irreducible polynomials. Use Kornblum's theorem and part a) to prove that

$$\gcd(\{\pi - A : \pi \equiv A \bmod M, \pi \notin S\}) = M$$

unless $p = 2$ and $M$ is relatively prime to either $T$ or $T + 1$ (or both), and then give a formula for the gcd in these cases as well. As in part b), your proof should work when $S$ is infinite with a suitable constraint.

(2) For a prime $p$, let $N_p = \#\{(x, y) \in \mathbf{Z}/p \times \mathbf{Z}/p : y^2 = x^3 - x\}$. Compute $N_p$ for $2 \le p \le 29$. Make some good observations, and try to prove some of them.

(What, you may ask, does this have to do with $L$-functions? Wait to find out in the solution set, or think carefully about your data and discover the connection for yourself.)

(3) Let $M$ be a nonzero polynomial in $\mathbf{F}_p[T]$, of degree $d > 1$. Let $\chi$ be a nontrivial Dirichlet character mod $M$. We know from class that $L(s, \chi)$ is a polynomial in $p^{-s}$ of degree $< d$, say

$$L(s, \chi) = \sum_{0 \le n \le d-1} a_n p^{-ns}.$$

As noted in class, this equation extends the definition of $L(s, \chi)$ to all real $s$. In particular, $L(0, \chi) = \sum_{n \le d-1} a_n$. Recalling how the coefficients $a_n$ are determined, we find

$$\sum_{n \le d-1} a_n = \sum_{n \le d-1} \sum_{\deg f = n} \chi(f) = \sum_{\deg f < d} \chi(f) = 0,$$

since the sum of a nontrivial character over a group is 0, the polynomials of degree less than $d = \deg M$ (which are relatively prime to $M$) represent all the units of the group $(\mathbf{F}_p[T]/M)^\times$, and $\chi(f) = 0$ if $f$ is a nonunit mod $M$. Thus $L(0,\chi) = 0$.

Alas, this is incorrect. We've seen examples in class where $L(s,\chi) = 1$ for all $s$, so in particular $L(0,\chi) = 1$. Where is the error in the above argument? (Do *not* give examples where the argument fails. Pinpoint the actual error in the "proof".)

(4) Let $p$ be an odd prime and $\omega$ a fixed nontrivial $p$th root of unity in $\mathbf{C}$, e.g., $\cos(2\pi/p) + i\sin(2\pi/p)$. For a monic polynomial

$$f(T) = T^n + c_{n-1}T^{n-1} + \cdots + c_0$$

in $\mathbf{F}_p[T]$, define $\chi(f) = (\frac{c_0}{p})\omega^{c_{n-1}}$, where $(\frac{\cdot}{p})$ is the Legendre symbol. (It is okay to to raise $\omega$ to an exponent taken from $\mathbf{F}_p = \mathbf{Z}/p$ since the exponent only matters modulo $p$ anyway.) In particular, $\chi(1) = 1$ and $\chi(T + c) = (\frac{c}{p})\omega^c$.

a) Show $\chi(fg) = \chi(f)\chi(g)$ for any two monic $f$ and $g$ in $\mathbf{F}_p[T]$.

b) Prove $\chi$ is *not* periodic, i.e., there is no polynomial $M(T) \in \mathbf{F}_p[T]$ such that $\chi(f) = \chi(g)$ when $f$ and $g$ are monic with $f \equiv g \bmod M$.

c) For $s > 1$, define

$$L(s,\chi) = \sum_{\text{monic } f} \frac{\chi(f)}{Nf^s}.$$

Show that $L(s,\chi) = 1 + a_1/p^s$, where

$$a_1 = \sum_{j=1}^{p-1} \left(\frac{j}{p}\right)\omega^j.$$

Warning: Since $\chi$ is not a Dirichlet character for $\mathbf{F}_p[T]$, be careful about appealing to results from class which were only proved for $L$-functions of Dirichlet characters.

(5) (An $L$-function for a quadratic modulus)

Let $p$ be an odd prime and fix a nontrivial $p$th root of unity $\omega$. Define a function $\chi\colon (\mathbf{F}_p[T]/T^2)^\times \to \mathbf{C}^\times$ by

$$\chi(c_0 + c_1T + \cdots + c_nT^n) = \left(\frac{c_0}{p}\right)\omega^{c_1/c_0}.$$

(The mod $p$ division in the exponent of $\omega$ makes sense since $c_0 \neq 0$ in $\mathbf{F}_p$ for a polynomial that is a unit mod $T^2$.) Note $\chi$ is defined for all units mod $T^2$, not just for the monic polynomials which are units mod $T^2$.

a) Show $\chi(fg) = \chi(f)\chi(g)$ for any two polynomials $f$ and $g$ that are units mod $T^2$.

b) Extend $\chi$ to the nonunits mod $T^2$ by setting it 0 there, and define the $L$-function of $\chi$, for $s > 1$, by

$$L(s,\chi) = \sum_{\text{monic } f} \frac{\chi(f)}{Nf^s}.$$

Show $L(s,\chi) = 1 + a_1/p^s$, where

$$a_1 = \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) \omega^j.$$

Comparing with the previous exercise, we see that the *same* $L$-function arises from a Dirichlet character mod $T^2$ and from a non-Dirichlet character.

In case you haven't seen a sum like $a_1$ before, it is called a *Gauss sum*. This Gauss sum can be used to give a proof of quadratic reciprocity which is less roundabout than the proof on the PROMYS sets, and it also arises in the more advanced study of the classical Dirichlet $L$-function $L(s, (\frac{\cdot}{p}))$.

(6) (An $L$-function for a cubic modulus)

a) Show 2 and $T+1$ generate the units of $\mathbf{F}_3[T]/(T^3+T)$.

b) Define a character $\chi$ by $\chi(2) = -1$ and $\chi(T+1) = -1$, extended by multiplicativity to other units. Compute $L(s,\chi)$ and the associated polynomial $P_\chi(x)$.

c) Same as part b), but let $\chi$ be determined by $\chi(2) = -1$ and $\chi(T+1) = i$.

(7) (An $L$-function for a quartic modulus)

a) Show $T^4 + T + 2$ is irreducible in $\mathbf{F}_3[T]$.

b) Show $T$ generates the units of $\mathbf{F}_3[T]/(T^4+T+2)$.

c) Define a character mod $T^4+T+2$ by $\chi(T) = i$. Compute $L(s,\chi)$ as a polynomial in $1/3^s$. (As a check on your work, the final answer should be a polynomial $P_\chi(x)$ which has a root at $x = 1$, and the other two roots have the same absolute value.)

(8) Fix nonzero $M \in \mathbf{F}_p[T]$ and integers $a$ and $b$. For $(A,M) = 1$, show there are infinitely many monic irreducible $\pi$ such that both $\pi \equiv A \bmod M$ and $\deg \pi \equiv a \bmod b$. Compute a density for such $\pi$. (Hint: In the spirit of the proof of Dirichlet's theorem, you want to get a good formula for the sum

$$\sum_{\substack{\pi \equiv A \bmod M \\ \deg \pi \equiv a \bmod b}} \frac{1}{N\pi^s},$$

where the sum is taken over monic irreducible $\pi$ satisfying the indicated conditions. For $b$th roots of unity $\omega$, consider the characters $\psi_{\chi,\omega}(f) = \omega^{\deg f}\chi(f)$ and the corresponding $L$-functions. Note

$$\frac{1}{b}\sum_{\omega^b=1} \omega^{d-a} = \begin{cases} 1, & \text{if } d \equiv a \bmod b, \\ 0, & \text{if } d \not\equiv a \bmod b, \end{cases}$$

so the condition of being congruent to $a$ mod $b$ can be expressed via a sum of $b$th roots of unity.)