ZETA AND L-FUNCTIONS
HOMEWORK 4
JULY 20, 2000

Due: Thursday, July 27 at the beginning of class

Problems:

(1) For $a, b > 0$ and $s > a + b$, show

$$\sum_{n \geq 1} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}.$$

The case $a = b = 0$ was Exercise 1 on Set 3.

Be sure your argument is written not just logically, but clearly. *Nobody* wants to read a convoluted argument, and you should aspire to do more than simply write solutions: write solutions well!

(2) Show the following statements are logically equivalent:

i) For each pair of relatively prime positive integers a and m, there is some prime $p \equiv a \bmod m$.

ii) For each pair of relatively prime positive integers a and m, there are infinitely many primes $p \equiv a \bmod m$.

Well, trivially ii) implies i). So just show i) implies ii). Your argument should not involve any complicated math at all.

(3) Let $\{a_n\}_{n\geq 1}$ be a totally multiplicative sequence which is periodic, say $a_{n+N} = a_n$ for some N and all $n \geq 1$. We do not assume that N is necessarily the minimal period.

Since $a_n = a_1 a_n$ for all n, $a_1 = a_1^2$, so $a_1 = 0$ or 1. To avoid the case when $a_n = 0$ for all n, we *must* take $a_1 = 1$. We also *assume* the sequence a_n is not identically 1: $a_n \neq 1$ for some n. In particular, the period N exceeds 1.

a) If $(n, N) = 1$, show $a_n^{\varphi(N)} = 1$. In particular, $|a_n| = 1$.

b) Show $a_N = 0$, and then $a_p = 0$ for some prime factor p of N.

c) If $d|N$ and $a_d \neq 0$, show $\{a_n\}$ has period N/d: $a_{n+N/d} = a_n$ for all $n \geq 1$.

d) If $d|N$ and $a_d \neq 0$, show a_d is a root of unity. (Warning: d and N/d need not be relatively prime.)

(4) The *Hurwitz partial zeta function* $\zeta(s, b)$, for $b > 0$, is defined by the series

$$\zeta(s,b) = \sum_{n\geq 0} \frac{1}{(n+b)^s} = \frac{1}{b^s} + \frac{1}{(1+b)^s} + \frac{1}{(2+b)^s} + \frac{1}{(3+b)^s} + \cdots.$$

Neglecting the first term and comparing the other terms with the terms in $\zeta(s)$, we see $\zeta(s, b)$ converges for $s > 1$. As an example, $\zeta(s, 1) = \zeta(s)$.

For $b \neq 1$, the partial zeta function does not admit an Euler product, so is a basic counterexample to my remark in class that "natural" zeta functions have Euler products. The partial zeta function can be used to prove facts about Dirichlet L-functions through part a) below, and many generalizations of $\zeta(s)$ admit "partial zeta" analogues that are useful.

a) For any mod m Dirichlet character χ, show for $s > 1$ that

$$L(s, \chi) = \frac{1}{m^s} \sum_{j=1}^{m} \chi(j)\zeta(s, j/m).$$

b) Use the estimate on y^α in the first technical handout to prove that

$$\frac{1}{s-1}\left(\frac{1}{b^{s-1}} - \frac{1}{(N+1+b)^{s-1}}\right) \leq \sum_{n=0}^{N} \frac{1}{(n+b)^s} \leq \frac{1}{b^s} + \frac{1}{s-1}\left(\frac{1}{b^{s-1}} - \frac{1}{(N+b)^{s-1}}\right).$$

When $b = 1$, the middle sum is $\sum_{n=1}^{N+1} n^{-s}$ and these inequalities reduce to what you proved (well, what you were asked to prove) in Exercise 2, Set 1.

Letting $N \to \infty$ in these inequalities yields the bounds

$$\frac{1}{(s-1)b^{s-1}} \leq \zeta(s, b) \leq \frac{1}{b^s} + \frac{1}{(s-1)b^{s-1}},$$

which generalizes our bounds on $\zeta(s)$.

(5) (Arithmetic functions on $\mathbf{F}_p[T]$ and $\mathbf{Z}[i]$)

a) Define functions $\mu_{\mathbf{F}_p[T]}$ and $\mu_{\mathbf{Z}[i]}$ on nonzero elements of $\mathbf{F}_p[T]$ and $\mathbf{Z}[i]$ such that the equations

$$\frac{1}{\zeta_{\mathbf{F}_p[T]}(s)} = \sum_{\text{monic } f} \frac{\mu_{\mathbf{F}_p[T]}(f)}{\mathrm{N}f^s}, \qquad \frac{1}{\zeta_{\mathbf{Z}[i]}(s)} = \sum_{(\alpha)} \frac{\mu_{\mathbf{Z}[i]}(\alpha)}{\mathrm{N}\alpha^s}$$

hold when $s > 1$. (Verify these equations only *after* giving your definition. You can't use these equations to define the Möbius functions because Dirichlet series indexed by $\mathbf{F}_p[T]$ or $\mathbf{Z}[i]$ do not have uniquely determined coefficients.)

b) Prove an analogue for these new Möbius functions of Möbius inversion.

c) For each integer $k \geq 0$, define on $\mathbf{F}_p[T]$ and on $\mathbf{Z}[i]$ an analogue of the function $\sigma_k(n) = \sum_{d|n, d>0} d^k$ so that the equation $\zeta(s)\zeta(s-k) = \sum \sigma_k(n)n^{-s}$ generalizes with $\zeta(s)$ replaced by the zeta functions of $\mathbf{F}_p[T]$ and $\mathbf{Z}[i]$. Taking $k = 0$ and $k = 1$ will give analogues of the elementary number theoretic functions $\tau(n)$ and $\sigma(n)$.

(6) Fix a complex number z_0 with $|z_0| = 1$ and $z_0 \neq 1$. (The number z_0 may or may not be a root of unity.) For a positive integer n, write its prime factorization as $n = p_1^{k_1} \cdots p_r^{k_r}$. Define

$$\psi(n) = z_0^{k_1 + \cdots + k_r}.$$

In particular, $\psi(1) = 1$ and $|\psi(n)| = 1$ for all n. We could write ψ as ψ_{z_0} if we want to indicate the dependence on z_0.

a) Show $\psi(nn') = \psi(n)\psi(n')$ for all positive integers n and n', so ψ is totally multiplicative.

b) The L-function of ψ is defined to be the Dirichlet series $L(s,\psi) = \sum_{n\geq 1}\psi(n)n^{-s}$. For which real s is it trivial to check the series converges? Give an Euler product for $L(s,\psi)$ and for $L(s,\psi\chi)$, where χ is a Dirichlet character. (The function $\psi\chi$ is just the product function, taking value $\psi(n)\chi(n)$ at the positive integer n.)

c) Show ψ is *not* periodic, i.e., an equation $\psi(n+m) = \psi(n)$ for some $m \geq 1$ and all $n \geq 1$ is impossible.

(7) In this exercise, we consider an L-function for a character on $\mathbf{Z}[i]$.

a) Show the units of $\mathbf{Z}[i]/(1+i)^3$ are represented by $\pm 1, \pm i$. (Since $(1+i)^3 = i(2+2i)$, we can also write the modulus $(1+i)^3$ as $2+2i$ without changing the resulting congruence class ring.)

b) Call a Gaussian integer *odd* if it is not divisible by $1+i$. Show a Gaussian integer is odd precisely when its norm is an odd integer in the usual sense.

c) Let α be an odd Gaussian integer, so $\alpha \bmod 2+2i$ is a unit. Let $\widetilde{\chi}(\alpha)$ be the unique unit from $\{\pm 1, \pm i\}$ such that $\alpha\widetilde{\chi}(\alpha) \equiv 1 \bmod 2{+}2i$. Compute $\widetilde{\chi}(\alpha)$ for $\alpha = 1+2i$, $2-3i$, $2+3i$, and $5+4i$, and check quite generally that on odd Gaussian integers $\widetilde{\chi}$ is totally multiplicative and commutes with complex conjugation: $\widetilde{\chi}(\alpha\beta) = \widetilde{\chi}(\alpha)\widetilde{\chi}(\beta)$ and $\widetilde{\chi}(\overline{\alpha}) = \overline{\widetilde{\chi}(\alpha)}$. Also show that $\widetilde{\chi}(\alpha)$ *changes* if α is replaced by a unit multiple other than α.

d) (This part has a rather long introduction.) We extend the definition of $\widetilde{\chi}$ to even Gaussian integers in the natural way: $\widetilde{\chi}(\alpha) = 0$ for even α. This extended function on all Gaussian integers is totally multiplicative and commutes with complex conjugation. (Don't bother writing out the proof of this.)

A complication arises now which we did not really see in our experience over $\mathbf{Z}$, and it is related to the fact that there is no natural way to pick out a multiplicatively stable family of nonassociate Gaussian integers (cf. Exercise 7, Set 2). Dirichlet series and Euler products over $\mathbf{Z}[i]$ are naturally indexed by Gaussian integers defined only up to multiplication by an arbitrary unit. Therefore the series $\sum_{(\alpha)}\widetilde{\chi}(\alpha)N\alpha^{-s}$ makes *no sense*, since $\widetilde{\chi}(\alpha)$ changes when α changes by a unit.

Quite generally, characters of $(\mathbf{Z}[i]/\alpha)^\times$ are sensitive to changes of the variable by a unit. Dirichlet characters can also have this property, e.g., $\chi_4(-n) = -\chi_4(n) \neq \chi_4(n)$ for odd n. However, over $\mathbf{Z}$ this doesn't lead to any problems because we *define* Dirichlet series and Euler products over $\mathbf{Z}$ as running over *positive* integers and *positive* primes. There's no such subterfuge in $\mathbf{Z}[i]$.

Here is the resolution, at least in the case of $\widetilde{\chi}$. For odd α, define $\chi_H(\alpha)$ to be the unique associate of α such that $\chi_H(\alpha) \equiv 1 \bmod 2+2i$. For even α, set $\chi_H(\alpha) = 0$. In terms of $\widetilde{\chi}$, $\chi_H(\alpha) = \widetilde{\chi}(\alpha)\alpha$. The function χ_H is called a *Hecke character* on $\mathbf{Z}[i]$, with modulus $2+2i$.

Now comes the actual problem to solve. Show the value of $\chi_H(\alpha)$ does not depend on the unit multiple of α and the corresponding *Hecke*

L-function

$$L(s, \chi_H) = \sum_{(\alpha)} \frac{\chi_H(\alpha)}{N\alpha^s}$$

converges for $s > 3/2$. When terms of $L(s, \chi_H)$ with like norm are collected together, so $L(s, \chi_H)$ is expressed as a Dirichlet series $\sum_{n \geq 1} a_n n^{-s}$, then this series converges on the larger interval $s > 5/6$, but this is pretty delicate. Don't try to prove it!

(Warning: $|\chi_H(\alpha)| = |\alpha|$, which usually is not 1, and χ_H is *not* periodic on $\mathbf{Z}[i]$. It is analogous to the function $\chi_4(n)n$ on $\mathbf{Z}$.)

e) Let $\overline{\chi}_H$ be the complex conjugate function of χ_H. Show $\overline{\chi}_H \neq \chi_H$ but $L(s, \overline{\chi}_H) = L(s, \chi_H)$ for $s > 3/2$. So unlike Dirichlet L-functions, which are different for different characters, L-functions of different (Hecke) characters on $\mathbf{Z}[i]$ may be equal.

f) Since χ_H is totally multiplicative, $L(s, \chi_H)$ admits an Euler product over $\mathbf{Z}[i]$. Collecting together the Euler factors corresponding to conjugate Gaussian primes, show $L(s, \chi_H)$ admits a quadratic Euler product over the primes in $\mathbf{Z}$,

$$L(s, \chi_H) = \prod_p \frac{1}{1 - a_p p^{-s} + b_p p^{-2s}},$$

for suitably large s (how large?). Explicitly relate a_p to arithmetic properties of p in $\mathbf{Z}[i]$. In particular, compute a_p and b_p for $2 \leq p \leq 29$.

g) Writing $L(s, \chi_H) = \sum_{n \geq 1} a_n n^{-s}$, determine a_n for $1 \leq n \leq 30$.

(8) (Extra) Here are two bonus exercises which are straightforward to solve with the complex logarithm. However, the statements themselves do not involve the complex logarithm. Try to prove these statements *without* using logarithms of nonreal complex numbers. (We used these two results in our proof of Dirichlet's theorem.)

a) Prove that for complex z with $|z| < 1$,

$$-\log|1 - z| = \sum_{n \geq 1} \frac{r^n \cos(n\theta)}{n},$$

where $z = r(\cos\theta + i \sin\theta)$.

b) Let c be a complex number with $0 \leq |c| \leq 1$. In the rational function

$$\frac{1}{(1-x)^2(1-cx)(1-\overline{c}x)} = \frac{1}{1-x} \cdot \frac{1}{1-x} \cdot \frac{1}{1-cx} \cdot \frac{1}{1-\overline{c}x},$$

expand each factor into a geometric series and multiply, getting a series $\sum b_n x^n$ that converges for $|x| < 1$. Prove $b_n \geq 0$ for all n.