# ZETA AND $L$-FUNCTIONS
# HOMEWORK 2
# JULY 10, 2000

*Due: Monday, July 17 at the beginning of class*

*Problems:*

(1) Write the letter $\chi$ properly 25 times, by hand. (Warning: This is *not* the letter $x$! Note $\chi$ hangs partly below the line of writing, like $j$.)

(2) Use the Euler product for $\zeta(s)$ and $L(s)$ to prove

$$\frac{1}{\zeta(s)} = \sum_{n\geq 1} \frac{\mu(n)}{n^s}, \quad \frac{1}{L(s)} = \sum_{n\geq 1} \frac{\mu(n)\chi_4(n)}{n^s}$$

for $s > 1$. Indicate which of the basic convergence theorems you use from the second technical handout.

(3) Let $a_1, a_2, \ldots$ be a totally multiplicative sequence (i.e., $a_{mn} = a_m a_n$ for all positive integers $m$ and $n$), with $|a_n| \leq 1$ for all $n$. Use appropriate convergence theorems from the second technical handout to prove that

$$\sum_{n\geq 1} \frac{a_n}{n^s} = \prod_p \frac{1}{1 - a_p p^{-s}}$$

for $s > 1$, where both product and sum can be rearranged.

(4) For an integer $n$, let

$$\chi_3(n) = \begin{cases} 1, & \text{if } n \equiv 1 \bmod 3, \\ -1, & \text{if } n \equiv 2 \bmod 3, \\ 0, & \text{if } n \equiv 0 \bmod 3. \end{cases}$$

Note $\chi_3(nn') = \chi_3(n)\chi_3(n')$ for all integers $n$ and $n'$. The series

$$\begin{aligned} L(s, \chi_3) &= \sum_{n\geq 1} \frac{\chi_3(n)}{n^s} \\ &= 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} - \frac{1}{8^s} + \ldots \end{aligned}$$

converges for $s > 0$, and for $s > 1$ we have

$$L(s, \chi_3) = \prod_p \frac{1}{1 - \chi_3(p)p^{-s}}.$$

a) Prove by analytic methods that there are infinitely many primes $p \equiv 1 \bmod 3$ and that there are infinitely many primes $p \equiv 2 \bmod 3$. Mimic the argument used to handle the case of primes mod 4.

b) Give an elementary Euclid-style proof of this infinitude.

(5) For a polynomial $f(T)$ in $\mathbf{F}_3[T]$, set

$$\chi(f) = \begin{cases} 1, & \text{if } f(0) = 1, \\ -1, & \text{if } f(0) = 2, \\ 0, & \text{if } f(0) = 0. \end{cases}$$

(Warning: Although $2 \equiv -1 \bmod 3$, it is not correct that $\chi(f) = f(0)$; $f(0)$ is in $\mathbf{F}_3$, while $\chi(f)$ is in the real numbers.)

a) Show $\chi(fg) = \chi(f)\chi(g)$ for all $f, g$ in $\mathbf{F}_3[T]$.

b) Define

$$L(s, \chi) = \sum_{\text{monic } f} \frac{\chi(f)}{Nf^s}.$$

Use appropriate convergence theorems to show this series converges when $s > 1$ and there is an Euler product

$$L(s, \chi) = \prod_{\text{monic } \pi} \frac{1}{1 - \chi(\pi)N\pi^{-s}},$$

where the order of addition and multiplication does not matter. (Note that unlike the case of $L(s)$ from class, the $L$-function $L(s, \chi)$ here is not initially defined for $0 < s \leq 1$.)

c) The functions $\chi_3$ and $\chi_4$ on $\mathbf{Z}$ are defined modulo 3 and 4, respectively. What should be the "modulus" for $\chi$ on $\mathbf{F}_3[T]$?

d) For $s > 1$, prove $L(s, \chi) = 1$. (!!)

e) Prove by analytic methods that in $\mathbf{F}_3[T]$, there are infinitely many monic irreducibles with constant term 1 and infinitely many monic irreducibles with constant term 2.

(6) For a Gaussian integer $\alpha = a + bi$, recall its norm is $N\alpha = a^2 + b^2$ and this norm is multiplicative by simple algebraic calculations.

a) For nonzero $\alpha$ in $\mathbf{Z}[i]$, prove $\#\mathbf{Z}[i]/\alpha = N\alpha$. (This part is logically not needed for the remaining parts of the problem. Its purpose is to show the familiar norm function on $\mathbf{Z}[i]$ can be thought of in the same combinatorial way as the norm on $\mathbf{F}_p[T]$ and the absolute value on $\mathbf{Z}$.)

b) Define the zeta function of $\mathbf{Z}[i]$ to be

$$\zeta_{\mathbf{Z}[i]}(s) = \prod_{(\pi)} \frac{1}{1 - N\pi^{-s}},$$

where $\prod_{(\pi)}$ designates a product over nonassociate irreducibles of $\mathbf{Z}[i]$. That is, out of every four associate irreducibles, one term is contributed to the product.

Prove the Euler product defining $\zeta_{\mathbf{Z}[i]}(s)$ converges for $s > 1$ and then show

$$\zeta_{\mathbf{Z}[i]}(s) = \sum_{(\alpha)} \frac{1}{N\alpha^s},$$

where $\sum_{(\alpha)}$ is a summation over nonassociate elements of $\mathbf{Z}[i]$. Explicitly cite any convergence theorems you use for infinite series and products, as well as any arithmetic properties you use of $\mathbf{Z}[i]$. (Hint: At most two nonassociate irreducible $\pi$ divide any given integer prime $p$, and $N\pi \geq p$.)

c) For $s > 1$, use the Euler product definition of $\zeta_{\mathbf{Z}[i]}(s)$ to prove

$$\log(\zeta_{\mathbf{Z}[i]}(s)) = \sum_{(\pi)} \frac{1}{N\pi^s} + g(s),$$

where $g(s)$ is a Dirichlet series with terms $\geq 0$ which converges for $s > 1/2$.

d) For $s > 1$, prove $\zeta_{\mathbf{Z}[i]}(s) = \zeta(s)L(s)$, where $L(s)$ is the $L$-function of $\chi_4$ from lecture. This analytic identity encodes the description of how primes in $\mathbf{Z}$ factor in $\mathbf{Z}[i]$, and could be used to give an alternate proof of part c).

(7) You may be wondering why we did not define $\zeta_{\mathbf{Z}[i]}(s)$ as a sum or product over individual elements, rather than over classes of associate elements. To give a defining formula over individual elements in any kind of natural way, we want $\mathbf{Z}[i]$ to have some notion corresponding to "positive" in $\mathbf{Z}$ and "monic" in $\mathbf{F}_p[T]$. At the very least, we need a subset $H \subset \mathbf{Z}[i] - \{0\}$ with the following two properties:

i) Each nonzero Gaussian integer has a unique unit multiple in $H$.

ii) $H$ is closed under multiplication.

Note that we only describe $H$ by multiplicative properties, not by additive properties.

It is natural to include the following additional condition:

iii) $\mathbf{Z}^+ \subset H$.

Show a set $H$ of nonzero Gaussian integers satisfying properties i), ii), and iii) does *not* exist, and describe an example of a set $H$ satisfying i) and ii), but not iii).

Hint: Think about the associates of $1 + i$ and $(1 + i)^2$.