

ZETA AND L-FUNCTIONS
HOMEWORK 1
JULY 3, 2000

Due: Monday, July 10 at the beginning of class

Problems:

- (1) Write the letter ζ properly 25 times, by hand. If you can't manage this task, you will have a lot of trouble in this course. (Warning: This is *not* the letter ξ !)
- (2) In this exercise, the estimate on y^α in the technical handout will be convenient.
 - a) Show by induction on N that for $s > 1$.

$$\frac{1}{s-1} - \frac{1}{(s-1)(N+1)^{s-1}} \leq \sum_{n=1}^N \frac{1}{n^s} \leq 1 + \frac{1}{s-1} - \frac{1}{(s-1)N^{s-1}}.$$

(It is best to handle the two inequalities separately. The left side can be weakened to $1/(s-1) - 1/(s-1)N^{s-1}$, and this is the form in which the left inequality will often be used; but do not prove it in this weaker form!)

- b) For $s > 1$ and n a positive integer, show $0 < \frac{1}{n^s} - \frac{1}{(n+1)^s} < \frac{s}{n^{s+1}}$.
- (3) (A prime race) Let

$$\pi_{1,4}(x) = \#\{p \leq x : p \text{ prime and } p \equiv 1 \pmod{4}\},$$

$$\pi_{3,4}(x) = \#\{p \leq x : p \text{ prime and } p \equiv 3 \pmod{4}\}.$$

Compute these functions for $1 \leq x \leq 100$. Any conjectures? Try a similar comparison between $\pi_{1,3}(x)$ and $\pi_{2,3}(x)$.

- (4) For $n \geq 2$, let $\{z_1, \dots, z_n\}$ be the n different n th roots of unity in the complex numbers. Show $z_1 + \dots + z_n = 0$ by an algebraic argument. Why does this vanishing make sense geometrically? (If you don't know where the roots of unity are located in the complex numbers, speak to a classmate who does.)
 - (5) For a nonzero polynomial $f(T) \in \mathbf{F}_p[T]$, define the *norm* of f to be

$$Nf = \#(\mathbf{F}_p[T]/f(T)) = p^{\deg f}.$$

For example, the nonzero constants have norm 1.

The norm on $\mathbf{F}_p[T]$ is the analogue of the absolute value $|m| = \#(\mathbf{Z}/m)$ for nonzero integers m . Note $N(f_1 f_2) = (Nf_1)(Nf_2)$, just as $|m_1 m_2| = |m_1| |m_2|$.

- a) For $n \geq 0$, compute $\#\{f : N(f) = p^n\}$ and $\#\{\text{monic } f : N(f) = p^n\}$.
 - b) Define the *zeta function* of $\mathbf{F}_p[T]$ to be the series

$$\zeta_{\mathbf{F}_p[T]}(s) = \sum_{\text{monic } f} \frac{1}{Nf^s}.$$

Note we write Nf^s for $(Nf)^s$.

Why is monic the right analogue of positive?

Show this series converges for $s > 1$ and sum the series exactly. This will yield a formula for $\zeta_{\mathbf{F}_p[T]}(s)$ which makes sense for all real $s \neq 1$, rather than only for $s > 1$.

c) How does $(s-1)\zeta_{\mathbf{F}_p[T]}(s)$ behave as s tends to 1 from the right?

d) Justify the Euler product

$$\zeta_{\mathbf{F}_p[T]}(s) = \prod_{\pi} \frac{1}{1 - N\pi^{-s}},$$

where the product is taken over *monic* irreducible π in $\mathbf{F}_p[T]$. (We write $N\pi^{-s}$ for $(N\pi)^{-s}$.)

e) Prove $\mathbf{F}_p[T]$ contains infinitely many irreducibles by using the zeta function of $\mathbf{F}_p[T]$.

(6) (Summation by parts) Let u_1, u_2, \dots and v_1, v_2, \dots be sequences.

a) Prove that for all integers $N \geq 2$,

$$\sum_{n=1}^N u_n(v_n - v_{n-1}) = u_N v_N - \sum_{n=1}^{N-1} v_n(u_{n+1} - u_n),$$

where we set v_0 in the $n = 1$ term on the left hand side.

In practice, summation by parts is applied to a sum $\sum a_n b_n$ when we have information about the partial sums $B_n = b_1 + \dots + b_n$, since summation by parts lets us write $\sum a_n b_n = \sum a_n(B_n - B_{n-1})$ in terms of $\sum B_n(a_{n+1} - a_n)$, so the understood term B_n appears directly in the n th term. (Those who know calculus will recognize summation by parts as a discrete analogue of integration by parts, and see the way summation by parts is usually used as analogous to the way integration by parts is used; consecutive differences are replaced by derivatives and partial sums are replaced by integrals.)

b) For a sequence a_1, a_2, \dots , suppose there is a constant C such that

$$|a_1 + \dots + a_n| \leq C$$

for all n . By judicious choice of u_n and v_n in part a), show $\sum a_n n^{-s}$ converges for all $s > 0$ and $|\sum a_n n^{-s}| \leq Cs\zeta(s+1)$.