SUMS OF SQUARES: SET 5

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1. Can you use the proof of Pfister's theorem to obtain an identity

$$(a^2 + b^2)(c^2 + d^2) = (?)^2 + (?)^2$$

which is not the usual identity?

2. What formula does Pfister's 4-square identity provide for the product

$$(x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2)$$

when we view the factors as 4-square sums with $x_4 = 0$ and $y_4 = 0$?

3. Let's look at sums of two squares in $\mathbf{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbf{Z}\}$.

a) For $\alpha = a + b\sqrt{2}$ in $\mathbf{Z}[\sqrt{2}]$, set $\overline{\alpha} = a - b\sqrt{2}$. Show $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$ and $\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}$.

b) If $\alpha \in \mathbf{Z}[\sqrt{2}]$ is a sum of squares, show $\overline{\alpha}$ is as well.

c) Show $1 + \sqrt{2}$ is *not* a sum of squares in $\mathbf{Z}[\sqrt{2}]$, even though $1 + \sqrt{2}$ is positive.

d) Show $3 + \sqrt{2}$ is *not* a sum of squares in $\mathbf{Z}[\sqrt{2}]$, even though $3 + \sqrt{2}$ and $3 - \sqrt{2}$ are positive.

e) Show $3 + \sqrt{2}$ is a sum of squares in $\mathbf{Q}[\sqrt{2}]$. What about $1 + \sqrt{2}$?

f) Try other examples in $\mathbf{Q}[\sqrt{2}]$. When do you think an element of $\mathbf{Q}[\sqrt{2}]$ is a sum of two squares in $\mathbf{Q}[\sqrt{2}]$?

4. Is every element of $\mathbf{Q}[i]$ a sum of squares in $\mathbf{Q}[i]$?

5. The "fifteen theorem" says $a_1x_1^2 + \cdots + a_rx_r^2$, with $a_i \in \mathbf{Z}^+$, represents all positive integers using $x_i \in \mathbf{Z}$ provided it represents the integers from 1 to 15. (The case $a = 1$ recovers Lagrange's 4-square theorem.) Use the fifteeen theorem to decide which of the polynomials $x^2 + y^2 + z^2 + aw^2$, for $2 \le a \le 10$, represent all positive integers.