SUMS OF SQUARES: SET 3

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1. Starting with the 2-square representation

$$261 = \left(\frac{1725}{221}\right)^2 + \left(\frac{3126}{221}\right)^2,$$

carry out the geometric argument from class until you get a 2-square representation $261 = a^2 + b^2$ with $a, b \in \mathbf{Z}$.

2. Determine which of the following fractions are sums of two squares in $\mathbf{Q}$: $\frac{9}{10}, \frac{62}{77}, \frac{369}{40}$.

3. Starting with the 3-square representation

$$13 = \left(\frac{18}{11}\right)^2 + \left(\frac{15}{11}\right)^2 + \left(\frac{32}{11}\right)^2,$$

carry out the geometric argument from class until you get a 3-square representation $13 = a^2 + b^2 + c^2$ with $a, b, c \in \mathbf{Z}$.

4. Determine which of the following integers are sums of 3 squares in $\mathbf{Z}$, using Legendre's theorem: 124, 983, 2005.

5. In $\mathbf{Q}[\sqrt{2}]$, compute $\frac{1-9\sqrt{2}}{5+3\sqrt{2}}$ in the form $a + b\sqrt{2}$ with $a, b \in \mathbf{Q}$.

6. Just using modular arithmetic, show for $a, b, c \in \mathbf{Z}$ that

$$a^2 + b^2 + c^2 \equiv 0 \bmod 8 \Longrightarrow a, b, c \text{ are all even.}$$

Conclude that for $n \in \mathbf{Z}^+$, if we write $n = 4^k n'$ with $k \geq 0$ and n' not divisible by 4 (*i.e.*, extract the largest power of 4 from n), then n is a sum of 3 squares in $\mathbf{Z}$ if and only if n' is a sum of 3 squares in $\mathbf{Z}$. (Don't appeal to results in class about sums of 3 squares in $\mathbf{Q}$!)

7. Show that the following conditions on a positive integer n are equivalent:
1) $n = x^2 + y^2 + z^2$ for some $x, y, z \in \mathbf{Z}$,
2) $x^2 + y^2 + z^2 - nw^2 = 0$ has a solution in $\mathbf{Z}$ other than $(0, 0, 0, 0)$,
3) $x^2 + y^2 + z^2 = 0$ has a solution in $\mathbf{Q}[\sqrt{-n}]$ other than $(0, 0, 0)$.

8. For the following values of n, find a solution to $-1 = x^2 + y^2$ in $\mathbf{Q}[\sqrt{-n}]$: 1, 3, 5, 6, 10, 14.

9. From each of the equations

$$83 = 3^2 + 5^2 + 7^2, \quad 83 = \left(\frac{7}{3}\right)^2 + \left(\frac{53}{15}\right)^2 + \left(\frac{121}{15}\right)^2,$$

derive a solution to $-1 = x^2 + y^2$ in $\mathbf{Q}[\sqrt{-83}]$.

10. Generalize the formula for primitive Pythagorean triples in $\mathbf{Z}$ to a formula for all "primitive" solutions to $f(t)^2 + g(t)^2 = h(t)^2$ in polynomials $f(t), g(t), h(t)$.