SUMS OF SQUARES: SET 2

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1. Obtain a rational parametrization of the unit circle $x^2 + y^2 = 1$ using lines through $(1, 0)$.

2. Obtain a rational parametrization of the circle $x^2 + y^2 = 2$ by two methods:
a) Rewrite the equation as $(\frac{x+y}{2})^2 + (\frac{x-y}{2})^2 = 1$ and use a rational parametrization of the unit circle.
b) Draw lines through the point $(1, 1)$.

3. Obtain a rational parametrization of the right branch of $x^2 - y^2 = 1$ using lines through $(1, 0)$. What about the left branch?

4. Show three perfect squares $a^2 < b^2 < c^2$ are in arithmetic progression if and only if $2b^2 = a^2+c^2$. Then use exercise 2 to find four triples of perfect squares in arithmetic progression.

5. Use different choices in the formula

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$

to explain the multiple representation $65 = 1^2 + 8^2 = 4^2 + 7^2$. (Hint: $65 = 5 \cdot 13$.) Find an integer that is a sum of two squares in three different ways.

6. A positive integer is a sum of two squares if and only if every prime number dividing it which is $\equiv 3 \bmod 4$ has even multiplicity as a factor. Use this to decide if 53, 86, 97, 351, and 2205 are sums of two squares.

7. (For those who have had calculus) A peculiar substitution for integrals is

$$u = \tan\left(\frac{t}{2}\right).$$

a) If $u = \tan(t/2)$, show

$$\cos t = \frac{1 - u^2}{1 + u^2}, \quad \sin t = \frac{2u}{1 + u^2}, \quad \mathrm{d}t = \frac{2\mathrm{d}u}{1 + u^2}.$$

Interpret these formulas geometrically.
b) Use this substitution to compute $\int \frac{\sin t + \cos t}{1 + \cos t}\mathrm{d}t$.