# SUMS OF SQUARES: SET 1

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1. Use the identity
$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$$
to express $205 = 5 \cdot 41$ as a sum of two squares in $\mathbf{Z}$.

2. Modulo 4, the only squares are 0 and 1:

| $a \bmod 4$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $a^2 \bmod 4$ | 0 | 1 | 0 | 1 |

Therefore $a^2+b^2 \equiv 0, 1,$ or $2 \bmod 4$, so
$$n \equiv 3 \bmod 4 \Longrightarrow n \neq a^2+b^2.$$
Use this idea to show
$$n \equiv 7 \bmod 8 \Longrightarrow n \neq a^2+b^2+c^2.$$

3. One positive integral solution to the equation
$$1^2 + 2^2 + \cdots + n^2 = m^2$$
is $(m,n) = (1,1)$. Can you find another solution in positive integers?

4. In $\mathbf{Q}$, write $\frac{1}{5}$ as a sum of two squares. (Hint: $\frac{1}{5} = \frac{5}{5^2}$.) Show the set of all nonzero sums of two squares in $\mathbf{Q}$ is closed under multiplication and division. Is $\frac{5}{2}$ such a number? What about $\frac{7}{5}$?