Math 5230 - Algebraic Number Theory (Fall 2012) Due in MSB 318

Problem Set 5 11/2/12 at 4 PM

Although it is usually simpler to prove a general fact than to prove numerous special cases of it, for a student the content of a mathematical theory is never larger than the set of examples that are throughly understood. V. Arnold

1. Verify that $\mathbf{Q}(\sqrt{-2})$, $\mathbf{Q}(\sqrt{5})$, $\mathbf{Q}(\sqrt{-3})$, and $\mathbf{Q}(\sqrt{-19})$ have class number 1. (As a safety check, in all cases the Kronecker bound is less than 11, but compute it in each case.)

2. Verify that $\mathbf{Q}(\sqrt{10})$ and $\mathbf{Q}(\sqrt{-10})$ have class number 2. (Kronecker bound is less than 20.)

3. Verify that $\mathbf{Q}(\sqrt{-17})$ has class group that is cyclic of order 4.

4. Verify that $\mathbf{Q}(\sqrt{-21})$ has class group that is a product of two groups of order 2.

5. (Optional) This exercise gives an example where isomorphic rings with the same fraction field have *nonisomorphic* integral closures in a quadratic extension of their fraction field.

   Let F be a field not of characteristic 2 and $f(X) \in F[X]$ be a nonconstant squarefree polynomial, so $F(X, \sqrt{f(X)})$ is a quadratic extension of $F(X)$. From Set 2, the integral closure of $F[X]$ in $F(X, \sqrt{f(X)})$ is $F[X, \sqrt{f(X)}]$. Abbreviate this integral closure as R. The residue rings $R/\mathfrak{a}$, for nonzero ideals $\mathfrak{a}$, are finite-dimensional F-vector spaces (analogue of residue rings of a ring of integers being finite).

   a) For a nonzero prime ideal $\mathfrak{p}$ in R, let $(\pi(X))$ be the nonzero prime lying under it in $F[X]$ (that is, $\mathfrak{p} \cap F[X] = (\pi(X))$), so $\mathfrak{p} | \pi(X)R$. Show that if $\dim_F(R/\mathfrak{p}) = 1$, then $\deg \pi(X) = 1$.

   b) For every $c \in F$, show the principal ideal $(X - c)$ factors in R as follows:

$$(X - c) = \begin{cases} \mathfrak{p}^2, & \text{if } f(c) = 0, \\ \mathfrak{p}\mathfrak{p}', & \text{if } f(c) = \square \text{ in } F^\times, \text{ with } \mathfrak{p} \neq \mathfrak{p}', \\ \mathfrak{p}, & \text{if } f(c) \neq \square \text{ in } F^\times. \end{cases}$$

   Also determine the F-dimension of the residue field at each of these prime ideals.

   c) Now we look at an example. In $\mathbf{F}_7[X]$, $X^4 + 3 = (X-3)(X-4)(X^2+2)$, which is squarefree. Set $L = \mathbf{F}_7(X, \sqrt{X^4+3})$. This is a quadratic extension of $\mathbf{F}_7(X)$ and the integral closure of $\mathbf{F}_7[X]$ in L is $\mathbf{F}_7[X, \sqrt{X^4+3}]$. Use parts a and b to determine the number of prime ideals in $\mathbf{F}_7[X, \sqrt{X^4+3}]$ whose residue field is $\mathbf{F}_7$.

   d) With L as in part c, let $X' = 1/X$. The rings $\mathbf{F}_7[X]$ and $\mathbf{F}_7[X']$ are not equal but are obviously isomorphic to each other and these two rings have the *same* fraction field: $\mathbf{F}_7(X) = \mathbf{F}_7(X')$. Compute the integral closure of $\mathbf{F}_7[X']$ in L and show the number of prime ideals with residue field $\mathbf{F}_7$ is not the same number as you found in part c. (Hint: To find the integral closure of $\mathbf{F}_7[X']$ in L, set $Y = \sqrt{X^4+3}$, $Y' = Y/X^2 = YX'^2$, and compute Y'^2.)

   Thus $\mathbf{F}_7[X]$ and $\mathbf{F}_7[X']$ are isomorphic and have the same fraction field but they have nonisomorphic integral closures in the field L.