Math 5230 - Algebraic Number Theory (Fall 2012) Due in MSB 318
Problem Set 3 10/5/12 at 4 PM

When general theory proves the existence of some construction, then doing it in terms of explicit coordinate expressions is a useful exercise that helps one to keep a grip on reality, [but] this should not however be allowed to obscure the fact that the theory is really designed to handle the complicated cases, when explicit computations will often not tell us anything. M. Reid

1. (Quadratic Orders)

   a) Show the discriminant of any order O in a quadratic field K is not a perfect square and is 0 or 1 mod 4. (Hint: $\operatorname{disc}(O) = [\mathcal{O}_K : O]^2 \operatorname{disc}(\mathcal{O}_K)$.)

   b) Show different quadratic orders – which could be in possibly different quadratic fields – have different discriminants. (Hint: Try to recover the data you need to write down the order from the formula for its discriminant.)

   c) For any integer D that is not a perfect square and satisfies $D \equiv 0$ or 1 mod 4, show there is a quadratic order with discriminant D. (This order is unique by part b.)

   d) Determine explicitly the quadratic orders with the following discriminants: -4, 45, 28, and -28. Make sure your answer is a ring and not just an additive group (e.g., it must contain 1).

2. (Dedekind's field, 1878)

   Let $K = \mathbf{Q}(\alpha)$, where $\alpha^3 - \alpha^2 - 2\alpha - 8 = 0$. Then $[K : \mathbf{Q}] = 3$ since $T^3 - T^2 - 2T - 8$ is irreducible mod 3.

   a) Verify $\operatorname{disc}(\mathbf{Z}[\alpha]) = -2012 = -4 \cdot 503$ by explicitly computing the trace-pairing matrix of the basis $\{1, \alpha, \alpha^2\}$ and its determinant.

   b) By part a, $\mathbf{Z}[\alpha] \subset \mathcal{O}_K \subset \frac{1}{2}\mathbf{Z}[\alpha]$. Use coset representatives for $\frac{1}{2}\mathbf{Z}[\alpha]/\mathbf{Z}[\alpha]$ to determine if $\mathcal{O}_K = \mathbf{Z}[\alpha]$ or not. If $\mathcal{O}_K \neq \mathbf{Z}[\alpha]$, find a $\mathbf{Z}$-basis for $\mathcal{O}_K$.

3. Compute a $\mathbf{Z}$-basis and the discriminant of the following cubic fields:

   a) $\mathbf{Q}(\alpha)$, $\alpha^3 - 10\alpha + 1 = 0$.

   b) $\mathbf{Q}(\alpha)$, $\alpha^3 + \alpha + 8 = 0$. (Hint: Stickelberger's theorem.)

   c) $\mathbf{Q}(\alpha)$, $\alpha^3 + \alpha^2 + 8 = 0$. (Hint: Stickelberger's theorem.)

4. From class, $\mathcal{O}_{\mathbf{Q}(i,\sqrt{-5})} = \mathbf{Z}[i] + \mathbf{Z}[i]\frac{i+\sqrt{-5}}{2}$. Show also that $\mathcal{O}_{\mathbf{Q}(i,\sqrt{-5})} = \mathbf{Z}[\sqrt{-5}] + \mathbf{Z}[\sqrt{-5}]\frac{i+\sqrt{-5}}{2}$. (**Note**: Since $\mathbf{Z}[\sqrt{-5}]$ is not a PID, *a priori* we have no reason to expect $\mathcal{O}_{\mathbf{Q}(i,\sqrt{-5})}$ is a free $\mathbf{Z}[\sqrt{-5}]$-module. Use bases over $\mathbf{Z}$ to make comparisons.)

5. For any field F, let $K = F(X, \alpha)$, where α is a root of $T^3 + XT + X$. This is irreducible in $K[T]$ since it is Eisenstein at X. Let R be the integral closure of $F[X]$ in K. We want to compute R.

a) Show the largest square factor of $\operatorname{disc}_{K/F(X)}(1, \alpha, \alpha^2)$ is X^2, so

$$F[X] + F[X]\alpha + F[X]\alpha^2 \subset R \subset \frac{1}{X}(F[X] + F[X]\alpha + F[X]\alpha^2).$$

The cases when F has characteristic 2 or 3 may need a separate consideration.

b) Show coset representatives for the additive group

$$\frac{1}{X}(F[X] + F[X]\alpha + F[X]\alpha^2)/(F[X] + F[X]\alpha + F[X]\alpha^2)$$

are $(a + b\alpha + c\alpha^2)/X$, where $a, b, c \in F$.

c) With notation as in part b, compute $\operatorname{Tr}_{K/F(X)}((a + b\alpha + c\alpha^2)/X)$ and conclude that if $(a + b\alpha + c\alpha^2)/X \in R$ then $a = 0$ as long as F does not have characteristic 3.

d) With notation as in part b, compute $\mathrm{N}_{K/F(X)}((a + b\alpha + c\alpha^2)/X)$ instead of $\operatorname{Tr}_{K/F(X)}((a + b\alpha + c\alpha^2)/X)$ and conclude that $(a + b\alpha + c\alpha^2)/X \in R$ only if $a = b = c = 0$, so $R = F[X][\alpha]$. (This bypasses part c and in particular is valid with no constraints on the characteristic of F.)