EXAMPLES OF MORDELL'S EQUATION

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1. Introduction

The equation $y^2 = x^3 + k$, for $k \in \mathbf{Z}$, is called Mordell's equation[1] on account of Mordell's long interest in it throughout his life. A natural number-theoretic task is the description of all rational and integral solutions to such an equation, either qualitatively (decide if there are finitely or infinitely many solutions in $\mathbf{Z}$ or $\mathbf{Q}$) or quantitatively (list or otherwise conveniently describe all such solutions). Mordell, in 1922, proved that for each $k \in \mathbf{Z}$, the equation $y^2 = x^3 + k$ has only finitely many integral solutions. The rational solutions present a different story: there may be finitely many or infinitely many, depending on the integer k. Whether or not there are infinitely many rational solutions is connected to one of the most outstanding open problems in number theory, the Birch and Swinnerton–Dyer conjecture.

Here we will describe all the integral solutions to Mordell's equation for some selected values of k, and make a few comments at the end about rational solutions.

2. Examples without Solutions

To prove $y^2 = x^3 + k$ has no integral solution for particular values of k, we will use congruence and quadratic residue considerations. Specifically, we will use the following descriptions of when -1, 2, and -2 are squares modulo an odd prime p:

$$\begin{aligned} -1 \equiv \square \bmod p &\iff p \equiv 1 \bmod 4, \\ 2 \equiv \square \bmod p &\iff p \equiv 1, 7 \bmod 8, \\ -2 \equiv \square \bmod p &\iff p \equiv 1, 3 \bmod 8. \end{aligned}$$

Our last example without solutions will not succumb to these kinds of methods.

Theorem 2.1. *The equation $y^2 = x^3 + 7$ has no integral solutions.*

Proof. (V. A. Lebesgue, 1869) Assume there is an integral solution. If x is even then $y^2 \equiv 7 \bmod 8$, but 7 mod 8 is not a square. Therefore x is odd. Rewrite $y^2 = x^3 + 7$ as

$$y^2 + 1 = x^3 + 8 = (x+2)(x^2 - 2x + 4). \tag{1}$$

The second factor, $x^2-2x+4 = (x-1)^2+3$, is positive. Since x is odd, $(x-1)^2+3 \equiv 3 \bmod 4$. Therefore $x^2 - 2x + 4$ is divisible by a prime $p \equiv 3 \bmod 4$ (otherwise all of its prime factors are 1 mod 4, but then that means $x^2-2x+4 \equiv 1 \bmod 4$, which is false). Since $p|(x^2-2x+4)$, we get $p|(y^2+1)$ from (1), so $y^2+1 \equiv 0 \bmod p$. Therefore $-1 \equiv \square \bmod p$, which contradicts $p \equiv 3 \bmod 4$.

Here's another approach, using the factor $x + 2$ instead of the factor $x^2 - 2x + 4$. Since (as seen above) x is odd and y is even, $x^3 \equiv x \bmod 4$ (true for any odd x), so reducing

[1] also Bachet's equation

$y^2 = x^3 + 7$ modulo 4 gives us $0 \equiv x + 3 \bmod 4$, so $x \equiv 1 \bmod 4$. Then $x + 2 \equiv 3 \bmod 4$. Moreover, $x+2 > 0$, since if $x \leq -2$ then $x^3 \leq -8$, so $x^3+7 \leq -1$, which contradicts x^3+7 being a perfect square. From $x+2$ being positive and congruent to 3 mod 4, it has a prime factor $p \equiv 3 \bmod 4$, so $y^2 + 1 \equiv 0 \bmod p$ from (1) and we get a contradiction as before. □

Theorem 2.2. *The equation $y^2 = x^3 - 5$ has no integral solutions.*

Proof. Assuming there is a solution, reduce modulo 4:

$$y^2 \equiv x^3 - 1 \bmod 4.$$

Here is a table of values of y^2 and $x^3 - 1$ modulo 4:

y	$y^2 \bmod 4$	x	$x^3 - 1 \bmod 4$
0	0	0	3
1	1	1	0
2	0	2	3
3	1	3	2

The only common value of y^2 mod 4 and $x^3 - 1$ mod 4 is 0, so y is even and $x \equiv 1 \bmod 4$. Then rewrite $y^2 = x^3 - 5$ as

$$y^2 + 4 = x^3 - 1 = (x - 1)(x^2 + x + 1). \tag{2}$$

Since $x \equiv 1 \bmod 4$, $x^2 + x + 1 \equiv 3 \bmod 4$, so $x^2 + x + 1$ is odd. Moreover, $x^2 + x + 1 = (x + 1/2)^2 + 3/4 > 0$, so $x^2 + x + 1 \geq 3$. Therefore $x^2 + x + 1$ must have a prime factor $p \equiv 3 \bmod 4$ (same reasoning as in the previous proof). Since p is a factor of $x^2 + x + 1$, p divides $y^2 + 4$ by (2), so $y^2 + 4 \equiv 0 \bmod p$. Therefore $-4 \equiv \square \bmod p$, so $-1 \equiv \square \bmod p$. This implies $p \equiv 1 \bmod 4$, contradicting $p \equiv 3 \bmod 4$. □

Our next two theorems will rely on the condition for when $2 \equiv \square \bmod p$.

Theorem 2.3. *The equation $y^2 = x^3 - 6$ has no integral solutions.*

Proof. Assume there is an integral solution. If x is even then $y^2 \equiv -6 \equiv 2 \bmod 8$, but 2 mod 8 is not a square. Therefore x is odd, so y is odd and $x^3 = y^2 + 6 \equiv 7 \bmod 8$. Also $x^3 \equiv x \bmod 8$ (true for any odd x), so $x \equiv 7 \bmod 8$.

Rewrite $y^2 = x^3 - 6$ as

$$y^2 - 2 = x^3 - 8 = (x - 2)(x^2 + 2x + 4), \tag{3}$$

with $x^2 + 2x + 4 \equiv 7^2 + 2 \cdot 7 + 4 \equiv 3 \bmod 8$. Since $x^2 + 2x + 4 = (x + 1)^2 + 3$ is positive, it must have a prime factor $p \equiv \pm 3 \bmod 8$ because if all of its prime factors are ± 1 mod 8 then $x^2 + 2x + 4 \equiv \pm 1 \bmod 8$, which is not true. Let p be a prime factor of $x^2 + 2x + 4$ with $p \equiv \pm 3 \bmod 8$. Since p divides $y^2 - 2$ by (3), we get $y^2 \equiv 2 \bmod p$. Thus $2 \equiv \square \bmod p$, so $p \equiv \pm 1 \bmod 8$, which is a contradiction.

We can get a contradiction using the factor $x-2$ also. Since $x \equiv 7 \bmod 8$, $x-2 \equiv 5 \bmod 8$. Also $x - 2 > 0$, since if $x \leq 2$ and $x - 2 \equiv 5 \bmod 8$ then $x \leq -1$, but then $x^3 - 6$ is negative so it can't be a perfect square. From $x - 2$ being positive and congruent to 5 mod 8, it has a prime factor $p \equiv \pm 3 \bmod 8$ and then $y^2 \equiv 2 \bmod p$ and we get a contradiction in the same way as before. □

Theorem 2.4. *The equation $y^2 = x^3 + 45$ has no integral solutions.*

Proof. Assume there is an integral solution. If y is odd then $x^3 = y^2 - 45 \equiv 1 - 45 \equiv 4 \bmod 8$, which is impossible. Therefore y is even, so x is odd. Reducing the equation mod 4, $0 \equiv x^3 + 1 \bmod 4$. Since $x^3 \equiv x \bmod 4$ for odd x, $x \equiv 3 \bmod 4$. Also, y is not a multiple of 3. If $3|y$ then the equation $y^2 = x^3 + 45$ shows 3 divides x. Write $x = 3x'$ and $y = 3y'$, so $9y'^2 = 27x'^3 + 45$, so $y'^2 = 3x'^3 + 5$, which implies $y'^2 \equiv 2 \bmod 3$, and that is impossible.

We will now take cases depending on whether $x \equiv 3 \bmod 8$ or $x \equiv 7 \bmod 8$. (If you know an elementary method which treats both cases in a uniform way, please tell me!)

Case 1: $x \equiv 3 \bmod 8$. Rewrite $y^2 = x^3 + 45$ as

$$y^2 - 72 = x^3 - 27 = (x-3)(x^2+3x+9). \tag{4}$$

The factor $x^2 + 3x + 9 = (x + 3/2)^2 + 27/4$ is positive and is congruent to 3 mod 8, so it has a prime factor $p \equiv \pm 3 \bmod 8$. Feeding this into (4),

$$y^2 \equiv 72 \equiv 2 \cdot 6^2 \bmod p. \tag{5}$$

We can't have $p = 3$ (just in case $p \equiv 3 \bmod 8$, this is something we need to deal with) since it would imply $y^2 \equiv 0 \bmod 3$, but we already checked y is not a multiple of 3. Since p is not 3, (5) implies $2 \equiv \square \bmod p$, so $p \equiv \pm 1 \bmod 8$, contradicting $p \equiv \pm 3 \bmod 8$.

Case 2: $x \equiv 7 \bmod 8$. Rewrite $y^2 = x^3 + 45$ as

$$y^2 - 18 = x^3 + 27 = (x+3)(x^2-3x+9). \tag{6}$$

The factor $x^2 - 3x + 9 = (x - 3/2)^2 + 27/4$ is positive and is congruent to 5 mod 8, so it has a prime factor $p \equiv \pm 3 \bmod 8$. From (6) we get $y^2 \equiv 18 \equiv 2 \cdot 3^2 \bmod p$. Arguing as in Case 1, we again find $p \equiv \pm 1 \bmod 8$, which is a contradiction. □

In our next example we will use the condition for when $-2 \equiv \square \bmod p$.

Theorem 2.5. *The equation $y^2 = x^3 + 46$ has no integral solutions.*

Proof. Assume there is an integral solution. If x is even then $y^2 \equiv 6 \bmod 8$, which has no solution, so x is odd and y is odd. Thus $y^2 \equiv 1 \bmod 8$ and $x^3 \equiv x \bmod 8$, so $1 \equiv x + 6 \bmod 8$, making $x \equiv 3 \bmod 8$.

Now rewrite $y^2 = x^3 + 46$ as

$$y^2 + 18 = x^3 + 64 = (x+4)(x^2-4x+16). \tag{7}$$

Since $x \equiv 3 \bmod 8$, the first factor on the right side of (7) is 7 mod 8 and the second factor is 5 mod 8. We will get a contradiction using either of these factors.

First we work with the quadratic factor $x^2 - 4x + 16 = (x-2)^2 + 12$, which is positive. Since it is 5 mod 8, it must have a prime factor p which is not 1 or 3 mod 8. Indeed, if all the prime factors of $x^2 - 4x + 16$ are 1 or 3 mod 8 then so is $x^2 - 4x + 16$, since $\{1, 3 \bmod 8\}$ is closed under multiplication. But $x^2 - 4x + 16 \not\equiv 1, 3 \bmod 8$. The prime p, not being 3 mod 8, is in particular not equal to 3. Also, $p \neq 2$ since $x^2 - 4x + 16$ is odd. Since $p|(x^2 - 4x + 16)$ we get by (7) that $p|(y^2 + 18)$, so $y^2 \equiv -18 \bmod p$. Hence $-18 \equiv \square \bmod p$, so $-2 \equiv \square \bmod p$. This implies $p \equiv 1$ or $3 \bmod 8$. But $p \not\equiv 1$ or $3 \bmod 8$, so we have a contradiction.

To get a contradiction using the factor $x + 4$, first let's check it is positive. There is no solution to $y^2 = x^3 + 46$ when y^2 is a perfect square less than 46 (just try $y^2 = 0, 1, 4, 9, 16, 25, 36$; there is no corresponding integral x), which means we must have $x^3 > 0$, so $x > 0$. Thus $x + 4 > 1$. Since $x + 4 \equiv 7 \bmod 8$, $x + 4$ must have a prime factor p which is not 1 or 3 mod 8, just as before. The prime p is not 2 since $x + 4$ is odd, and $p \neq 3$ since $p \not\equiv 3 \bmod 8$. Then $y^2 \equiv -18 \bmod p$ from (7) and we get a contradiction as before. □

Our final example that has no integral solutions is going to use more involved ideas than congruences.

Theorem 2.6. *The equation $y^2 = x^3 + 6$ has no integral solutions.*

Proof. First we do a divisibility check by 2 and 3. If x is even then $y^2 \equiv 6 \bmod 8$, which has no solution, so x is odd and then y is odd. If $3|y$ then $x^3 \equiv -6 \equiv 3 \bmod 9$, but the cubes mod 9 are 0, 1, and -1. So y is not divisible by 3.

Since $y^2 = x^3 + 6$ resembles $y^2 = x^3 - 6$, it is natural to try to adapt the congruence arguments as in Theorem 2.3. Begin by writing $y^2 + 2 = x^3 + 8 = (x+2)(x^2 - 2x + 4)$ and use the parity conditions on x and y to show $x \equiv 3 \bmod 8$. Alas, the kind of contradiction that arises in the proof of Theorem 2.3 doesn't show up in this new example. (Try it to see why!) Rather than search for other congruence methods, we will use unique factorization in $\mathbf{Z}[\sqrt{6}]$. The argument is taken from [2, pp. 22–23].

In $\mathbf{Z}[\sqrt{6}]$ write

$$x^3 = y^2 - 6 = (y + \sqrt{6})(y - \sqrt{6}). \tag{8}$$

Let's show $y + \sqrt{6}$ and $y - \sqrt{6}$ are relatively prime in $\mathbf{Z}[\sqrt{6}]$. Suppose there is a common divisor δ:

$$\delta | (y + \sqrt{6}), \quad \delta | (y - \sqrt{6}).$$

Taking the norm of either divisibility relation, $\mathrm{N}(\delta)$ divides $y^2 - 6$ in $\mathbf{Z}$, and $y^2 - 6$ is odd since y is odd, so $\mathrm{N}(\delta)$ is odd. Subtracting the divisibility relations, $\delta | 2\sqrt{6}$, so $\mathrm{N}(\delta)$ divides 24. Since $\mathrm{N}(\delta)$ is odd, $\mathrm{N}(\delta)$ is ± 1 or ± 3. If the norm is ± 3 then $3|(y^2 - 6)$, so $3|y$, a contradiction. Therefore δ has norm ± 1, so δ is a unit.

In (8), a product of relatively prime numbers in $\mathbf{Z}[\sqrt{6}]$ is a cube, so $y + \sqrt{6} = u(a + b\sqrt{6})^3$ where u is a unit and a and b are integers. The units in $\mathbf{Z}[\sqrt{6}]$ are $\pm(5 + 2\sqrt{6})^{\mathbf{Z}}$, and a unit cube can be absorbed into $(a + b\sqrt{6})^3$, so there are three possibilities:

$$y + \sqrt{6} = (a + b\sqrt{6})^3 \text{ or } y + \sqrt{6} = (5 + 2\sqrt{6})(a + b\sqrt{6})^3 \text{ or } y + \sqrt{6} = (5 + 2\sqrt{6})^2(a + b\sqrt{6})^3.$$

In the first case, equating the coefficients of $\sqrt{6}$ on both sides gives $1 = 3a^2b + 6b^3$, which is impossible. In the second case, equating the coefficients of $\sqrt{6}$ on both sides gives

$$1 = 5(3a^2b + 6b^3) + 2(a^3 + 18ab^2). \tag{9}$$

We will show this equation in a and b has no integral solution. Reducing both sides of (9) modulo 3, $1 \equiv 2a^3 \equiv 2a \bmod 3$, so $a \equiv 2 \bmod 3$. Therefore $a^2 \equiv 1 \bmod 3$ and $a^3 \equiv 8 \bmod 9$. Reducing both sides of (9) modulo 9, we now have

$$1 \equiv 5(3b + 6b^3) + 2(8) \bmod 9 \Rightarrow 0 \equiv 6b + 3b^3 + 6 \bmod 9 \Rightarrow 0 \equiv 2b + b^3 + 2 \bmod 3.$$

Since $2b + b^3 \equiv 0 \bmod 3$ no matter what b is, we have $0 \equiv 2 \bmod 3$, a contradiction. In the third case, we will reduce ourselves back to the second case. Multiply both sides by $5 + 2\sqrt{6}$ and absorb the $(5 + 2\sqrt{6})^3$ on the right into $(a + b\sqrt{6})^3$:

$$(5 + 2\sqrt{6})(y + \sqrt{6}) = (a + b\sqrt{6})^3.$$

Multiply by $(5 + 2\sqrt{6})^{-1} = 5 - 2\sqrt{6}$:

$$y + \sqrt{6} = (5 - 2\sqrt{6})(a + b\sqrt{6})^3.$$

Conjugate both sides:

$$y - \sqrt{6} = (5 + 2\sqrt{6})(a - b\sqrt{6})^3.$$

This is almost like the second case. Equating the coefficients of $\sqrt{6}$ on both sides,

$$-1 = 5(3a^2(-b) + 6(-b)^3) + 2(a^3 + 18ab^2).$$

Negating both sides,

$$1 = 5(3a^2b + 6b^3) + 2((-a)^3 + 18(-a)b^2), \tag{10}$$

which matches (9) with $-a$ in place of a. Since (9) has no integral solution, (10) has no integral solution either. □

3. Examples with Solutions

We will now look at some instances of Mordell's equation which have integral solutions. The goal in each case is to find all integral solutions. The main tool we will use is unique factorization (in different settings), and after some successes we will see that this technique eventually runs into difficulties.

We start with the case $k = 16$: the equation $y^2 = x^3 + 16$. There are two obvious integral solutions: $(x, y) = (0, \pm 4)$. A numerical search does not reveal additional integral solutions, so one might guess[2] that $(0, 4)$ and $(0, -4)$ are the only integral solutions. To prove this, we will use unique factorization in $\mathbf{Z}$.

Theorem 3.1. *The only integral solutions to $y^2 = x^3 + 16$ are $(x, y) = (0, \pm 4)$.*

Proof. First we determine the parity of an integral solution. Rewrite the equation as $x^3 = y^2 - 16 = (y + 4)(y - 4)$. If y is odd then $(y + 4, y - 4) = 1$ (why?), so both $y + 4$ and $y - 4$ are cubes because their product is a cube. They differ by 8, and no odd cubes differ by 8. Hence y is even, so x is even.

The right side of $y^2 = x^3 + 16$ is divisible by 8, so $4|y$. Writing $y = 4y'$, $16y'^2 = x^3 + 16$. Therefore $4|x$. Write $x = 4x'$, so $y'^2 = 4x'^3 + 1$, showing y' is odd. Write $y' = 2m + 1$, so $m^2 + m = x'^3$. Since $m^2 + m = m(m + 1)$ and $(m, m + 1) = 1$, both m and $m + 1$ are cubes. The only consecutive cubes are among $\{-1, 0, 1\}$, so m or $m + 1$ is 0. Therefore $x' = 0$, so $x = 0$ and $y = \pm 4$. □

Theorem 3.2. *The only $x, y \in \mathbf{Z}$ satisfying $y^2 = x^3 - 1$ is $(x, y) = (1, 0)$.*

Proof. First we check the parity of an integral solution. Suppose x is even, so $y^2 + 1 = x^3 \equiv 0 \bmod 8$. Then $y^2 \equiv -1 \bmod 8$. But $-1 \bmod 8$ is not a square. We have a contradiction, so x is odd, which means y has to be even.

Write the equation $y^2 = x^3 - 1$ as

$$x^3 = y^2 + 1,$$

which in $\mathbf{Z}[i]$ has the factored form

$$x^3 = (y + i)(y - i). \tag{11}$$

If the two factors on the right side are relatively prime in $\mathbf{Z}[i]$, then since their product is a cube, each factor must be a cube up to unit multiple, by unique factorization in $\mathbf{Z}[i]$. Moreover, since every unit in $\mathbf{Z}[i]$ is a cube ($1 = 1^3$, $-1 = (-1)^3$, $i = (-i)^3$, $-i = i^3$), unit factors can be absorbed into the cubes. Thus, provided we show $y + i$ and $y - i$ are relatively prime, (11) tells us $y + i$ and $y - i$ are themselves cubes.

[2]It's a tricky business to decide when to stop searching: $y^2 = x^3 + 24$ has integral solutions at $x = -2, 1, 10$, and 8158 (and no others).

To see that $y+i$ and $y-i$ are relatively prime, let δ be a common divisor. Since δ divides $(y+i)-(y-i) = 2i$, $\mathrm{N}(\delta)$ divides $\mathrm{N}(2i) = 4$. Also $\mathrm{N}(\delta)$ divides $\mathrm{N}(y+i) = y^2+1 = x^3$, which is odd. Therefore $\mathrm{N}(\delta)$ divides 4 and is odd, which means $\mathrm{N}(\delta) = 1$, so δ is a unit.

Now that we know $y+i$ and $y-i$ are relatively prime, we must have (as argued already)

$$y + i = (m + ni)^3$$

for some $m, n \in \mathbf{Z}$. Expanding the cube and equating real and imaginary parts,

$$y = m^3 - 3mn^2 = m(m^2 - 3n^2), \quad 1 = 3m^2n - n^3 = n(3m^2 - n^2).$$

The equation on the right tells us $n = \pm 1$. If $n = 1$, then $1 = 3m^2 - 1$, so $3m^2 = 2$, which has no integer solution. If $n = -1$, then $1 = -(3m^2 - 1)$, so $m = 0$. Therefore $y = 0$, so $x^3 = y^2 + 1 = 1$. Thus $x = 1$. □

Theorem 3.3. *The only $x, y \in \mathbf{Z}$ satisfying $y^2 = x^3 - 4$ are $(x, y) = (2, \pm 2)$ and $(5, \pm 11)$.*

Proof. We rewrite $y^2 = x^3 - 4$ in $\mathbf{Z}[i]$ as

$$x^3 = y^2 + 4 = (y + 2i)(y - 2i). \tag{12}$$

We will show that both factors on the right are cubes. Let's first see why this leads to the desired integral solutions. Write

$$y + 2i = (m + ni)^3$$

for some $m, n \in \mathbf{Z}$. Equating real and imaginary parts,

$$y = m(m^2 - 3n^2), \quad 2 = n(3m^2 - n^2).$$

From the second equation, $n = \pm 1$ or $n = \pm 2$. In each case we try to solve for m in $\mathbf{Z}$. The cases which work out are $n = 1$ and $m = \pm 1$, and $n = -2$ and $m = \pm 1$. In the first case, $y = \pm(1 - 3) = \pm 2$ and $x = 2$, while in the second case $y = \pm(1 - 3 \cdot 4) = \pm 11$ and $x = 5$.

It remains to show in (12) that $y + 2i$ and $y - 2i$ are cubes. Since $y^2 \equiv x^3 \bmod 2$ either x and y are both even or they are both odd. We will consider these cases separately, since they affect the greatest common factor of $y + 2i$ and $y - 2i$.

First suppose x and y are both odd. We will show $y + 2i$ and $y - 2i$ are relatively prime in $\mathbf{Z}[i]$. Let δ be a common divisor, so δ divides $(y + 2i) - (y - 2i) = 4i$. Therefore $\mathrm{N}(\delta)$ divides $\mathrm{N}(4i) = 16$. Since $\mathrm{N}(\delta)$ also divides $\mathrm{N}(y + 2i) = y^2 + 4 = x^3$, which is odd, we must have $\mathrm{N}(\delta) = 1$, so δ is a unit. This means $y + 2i$ and $y - 2i$ are relatively prime, so since their product in (12) is a cube and any unit in $\mathbf{Z}[i]$ are cubes, $y + 2i$ and $y - 2i$ are both cubes.

Now suppose x and y are both even. Write $x = 2x'$ and $y = 2y'$, so $4y'^2 = 8x'^3 - 4$. Dividing by 4, $y'^2 = 2x'^3 - 1$. Therefore y' is odd. We must have x' odd too, as otherwise $y'^2 \equiv -1 \bmod 4$, but $-1 \bmod 4$ is not a square. Writing

$$2x'^3 = y'^2 + 1 = (y' + i)(y' - i),$$

the factors on the right each have even norm, so each is divisible by $1 + i$. Divide the equation by $(1 + i)^2 = 2i$:

$$-ix'^3 = \frac{y' + i}{1 + i}\frac{y' - i}{1 + i}.$$

We will show the two factors on the right are relatively prime. Their difference is $2i/(1+i) = 1 + i$, so any common divisor has norm dividing $\mathrm{N}(1 + i) = 2$. Also any common divisor divides x'^3, so the norm divides $\mathrm{N}(x'^3) = x'^6$, which is odd. Thus any common divisor

of $(y'+i)/(1+i)$ and $(y'-i)/(1+i)$ has norm 1, so is a unit. As before, we now know $(y'+i)/(1+i)$ is a cube, so

$$y+2i = 2(y'+i) = -i(1+i)^2(y'+i) = i^3(1+i)^3\frac{y'+i}{1+i}$$

is a cube in $\mathbf{Z}[i]$. Similarly, $y-2i$ is a cube. □

Using unique factorization in another imaginary quadratic ring, we can find all the integral solutions to another case of Mordell's equation.

Theorem 3.4 (Fermat)**.** *The only integral solutions to* $y^2 = x^3 - 2$ *are* $(x,y) = (3,\pm 5)$.

Proof. (Euler) Suppose $y^2 = x^3 - 2$ with integral x and y. As in the previous proof, first we do a parity check on x and y. If x is even then $y^2 \equiv -2 \bmod 8$, but $-2 \bmod 8$ is not a square. Therefore x is odd, so y is also odd.

Write the relation between x and y as

$$x^3 = y^2 + 2.$$

In $\mathbf{Z}[\sqrt{-2}]$, we can rewrite this as

$$x^3 = (y+\sqrt{-2})(y-\sqrt{-2}). \tag{13}$$

The two factors on the right are relatively prime. Indeed, let δ be a common divisor, so δ divides their difference $(y+\sqrt{-2}) - (y-\sqrt{-2}) = 2\sqrt{-2}$, which means $\mathrm{N}(\delta)$ divides $\mathrm{N}(2\sqrt{-2}) = 8$. At the same time, $\mathrm{N}(\delta)$ divides $\mathrm{N}(y+\sqrt{-2}) = y^2+2$, which is odd since y is odd. So $\mathrm{N}(\delta)$ must be 1, which means δ is a unit in $\mathbf{Z}[\sqrt{-2}]$, so $y+\sqrt{-2}$ and $y-\sqrt{-2}$ are relatively prime in $\mathbf{Z}[\sqrt{-2}]$. From (13) and unique factorization in $\mathbf{Z}[\sqrt{-2}]$, $y+\sqrt{-2}$ and $y-\sqrt{-2}$ are both cubes up to unit multiple. The units in $\mathbf{Z}[\sqrt{-2}]$ are ± 1, which are both cubes, and therefore a unit multiple of a cube is also a cube. Hence $y+\sqrt{-2}$ and $y-\sqrt{-2}$ are both cubes.

Write

$$y+\sqrt{-2} = (m+n\sqrt{-2})^3$$

for some $m, n \in \mathbf{Z}$. It follows that

$$y = m^3 - 6mn^2 = m(m^2-6n^2), \quad 1 = 3m^2n - 2n^3 = n(3m^2-2n^2).$$

From the second equation, $n = \pm 1$. When $n = 1$ the second equation says $1 = 3m^2 - 2$, so $m = \pm 1$. Then $y = \pm 1(1-6) = \pm 5$ and $x^3 = y^2+2 = 27$, so we recover the solutions $(x,y) = (3,\pm 5)$. When $n = -1$ we have $1 = -(3m^2 - 2\cdot 1^2) = -(3m^2-2)$, so $1 = 3m^2$, which has no solution in $\mathbf{Z}$. □

Our treatment of $y^2 = x^3+16$, $y^2 = x^3-1$, $y^2 = x^3-4$, and $y^2 = x^3-2$ relied on features of $\mathbf{Z}$, $\mathbf{Z}[i]$, $\mathbf{Z}[\sqrt{-2}]$: they satisfy unique factorization and every unit in them is a cube. We can try the same technique on $y^2 = x^3+k$ with other values of k. The next three examples illustrate some new features.

Example 3.5. Consider Mordell's equation with $k = 1$: $y^2 = x^3+1$. There are several obvious integral solutions:

$$(x,y) = (-1,0), (0,\pm 1), \text{ and } (2,\pm 3).$$

We will use unique factorization in $\mathbf{Z}$ to try to show these are the only integral solutions. This will need a lot more work than Theorem 3.1, where we previously used unique factorization in $\mathbf{Z}$ to study $y^2 = x^3+16$.

We rewrite the equation in the form

$$x^3 = y^2 - 1 = (y+1)(y-1).$$

The integers $y+1$ and $y-1$ differ by 2, so $(y+1, y-1)$ is either 1 or 2.

First suppose y is even. Then $y+1$ and $y-1$ are both odd, so $(y+1, y-1) = 1$. (That is, any two consecutive odd integers are relatively prime.) Since $y+1$ and $y-1$ have a product which is a cube and they are relatively prime, unique factorization in $\mathbf{Z}$ tells us that they are both cubes or both the negatives of cubes. The negative of a cube is also a cube (since $-1 = (-1)^3$), so $y+1$ and $y-1$ are both cubes:

$$y + 1 = a^3, \quad y - 1 = b^3.$$

Subtracting, we have $a^3 - b^3 = 2$. Considering how cubes spread apart, the only cubes which differ by 2 are 1 and -1. So $a^3 = 1$ and $b^3 = -1$, meaning $a = 1$ and $b = -1$. Therefore $y + 1 = 1$, so $y = 0$ and $x = -1$. The integral solution $(-1, 0)$ of $y^2 = x^3 - 1$ is the only one where y is even.

Now suppose y is odd, so x is even. We expect to show that the only such integral solutions are $(0, \pm 1)$ and $(2, \pm 3)$. Since $y+1$ and $y-1$ are both even and differ by 2, $(y+1, y-1) = 2$. Either $y \equiv 1 \bmod 4$ or $y \equiv 3 \bmod 4$. Since (x, y) is a solution if and only if $(x, -y)$ is a solution, by negating y if necessary we may assume $y \equiv 1 \bmod 4$. Then $y + 1 \equiv 2 \bmod 4$ and $y - 1 \equiv 0 \bmod 4$. Dividing the equation $x^3 = y^2 - 1$ by 8, we have

$$\left(\frac{x}{2}\right)^3 = \frac{y+1}{2} \cdot \frac{y-1}{4}.$$

The two factors on the right are relatively prime, since $y+1$ and $y-1$ have greatest common factor 2 and we have divided each of them by a multiple of 2. Since the product of $(y-1)/2$ and $(y+1)/4$ is a cube and the factors are relatively prime, each of them is a cube:

$$\frac{y+1}{2} = a^3, \quad \frac{y-1}{4} = b^3$$

with integers a and b. (Actually, at first we can say simply that $(y+1)/2$ and $(y-1)/4$ are cubes up to sign, but $-1 = (-1)^3$ so we can absorb a sign into a and b if signs occur.) Solving each equation for y,

$$2a^3 - 1 = y = 4b^3 + 1, \tag{14}$$

so $a^3 - 2b^3 = 1$. We can spot right away two integral solutions to $a^3 - 2b^3 = 1$: $(a, b) = (1, 0)$ and $(a, b) = (-1, -1)$. In the first case, using (14) we get $y = 1$ (so $x = 0$) and in the second case we get $y = -3$ (so $x = 2$). We have found two integral solutions to $y^2 = x^3 + 1$ when $y \equiv 1 \bmod 4$: $(0, 1)$ and $(2, -3)$. Negating y produces the two solutions $(0, -1)$ and $(2, 3)$ where $y \equiv 3 \bmod 4$.

Any integral solution of $a^3 - 2b^3 = 1$ leads to the integral solution $(x, y) = (2ab, 4b^3 + 1)$ of $y^2 = x^3 + 1$, so showing the equation $y^2 = x^3 + 1$ has no further integral solutions is tantamount to showing $a^3 - 2b^3 = 1$ has no integral solution besides the two we already found, $(1, 0)$ and $(-1, -1)$. To study $a^3 - 2b^3 = 1$ introduces a whole new bag of complications, so we will simply stop and leave this matter unsettled.

Example 3.6. Consider $y^2 = x^3 - 5$. We have already seen in Theorem 2.2 that this equation has no integral solutions by a method that only uses calculations in $\mathbf{Z}$. Let's try to show there are no integral solutions using factorizations in $\mathbf{Z}[\sqrt{-5}]$.

We start with a parity check. If x is even then $y^2 \equiv -5 \equiv 3 \bmod 8$, but 3 mod 8 is not a square. Therefore x is odd, so y is even.

Write the equation as

$$x^3 = y^2 + 5 = (y + \sqrt{-5})(y - \sqrt{-5}).$$

Suppose δ is a common factor of $y + \sqrt{-5}$ and $y - \sqrt{-5}$. First of all, $\mathrm{N}(\delta)$ divides $y^2 + 5$, which is odd. Second of all, since δ divides $(y + \sqrt{-5}) - (y - \sqrt{-5}) = 2\sqrt{-5}$, $\mathrm{N}(\delta)$ divides $\mathrm{N}(2\sqrt{-5}) = 20$. Therefore $\mathrm{N}(\delta)$ is 1 or 5. If $\mathrm{N}(\delta) = 5$ then $5|(y^2 + 5)$, so $5|y$. Then $x^3 = y^2 + 5 \equiv 0 \bmod 5$, so $x \equiv 0 \bmod 5$. Now x and y are both multiples of 5, so $5 = x^3 - y^2$ is a multiple of 25, a contradiction. Hence $\mathrm{N}(\delta) = 1$, so δ is a unit. This shows $y + \sqrt{-5}$ and $y - \sqrt{-5}$ have no common factor in $\mathbf{Z}[\sqrt{-5}]$ except for units.

Since $y + \sqrt{-5}$ and $y - \sqrt{-5}$ are relatively prime and their product is a cube, they are both cubes (the units in $\mathbf{Z}[\sqrt{-5}]$ are ± 1, which are both cubes). Thus

$$y + \sqrt{-5} = (m + n\sqrt{-5})^3$$

for some integers m and n, so

$$y = m^3 - 15mn^2 = m(m^2 - 15n^2), \quad 1 = 3m^2n - 5n^3 = n(3m^2 - 5n^2).$$

From the second equation, $n = \pm 1$. If $n = 1$ then $1 = 3m^2 - 5$, so $3m^2 = 6$, which has no integral solution. If $n = -1$ then $1 = -(3m^2 - 5)$, so $3m^2 = 4$, which also has no integral solution. We appear to have recovered the fact that $y^2 = x^3 - 5$ has no integral solutions.

Alas, there is an error in Example 3.6. When we wrote certain numbers in $\mathbf{Z}[\sqrt{-5}]$ as cubes, we were implicitly appealing to unique factorization in $\mathbf{Z}[\sqrt{-5}]$, which is in fact false. A counterexample to unique factorization in $\mathbf{Z}[\sqrt{-5}]$ is $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. That doesn't mean the numbers in $\mathbf{Z}[\sqrt{-5}]$ which we wanted to be cubes might not be cubes, but our justification for those conclusions is certainly faulty. It *is* true in $\mathbf{Z}[\sqrt{-5}]$ that if $\alpha\beta$ is a cube and α and β only have common factors ± 1 then α and β are both cubes, but to explain why requires additional techniques to circumvent the lack of unique factorization.

Example 3.7. Consider $y^2 = x^3 - 26$. Two obvious integral solutions are $(3, \pm 1)$. Let's use factorizations in $\mathbf{Z}[\sqrt{-26}]$ to see if $(3, \pm 1)$ are the only integral solutions.

If x is even then $y^2 \equiv -26 \equiv 6 \bmod 8$, but 6 mod 8 is not a square. Therefore x is odd, so y is odd too.

Rewrite the equation as

$$x^3 = y^2 + 26 = (y + \sqrt{-26})(y - \sqrt{-26}).$$

Let δ be a common factor of $y + \sqrt{-26}$ and $y - \sqrt{-26}$ in $\mathbf{Z}[\sqrt{-26}]$. Then $\mathrm{N}(\delta)$ divides $y^2 + 26$, which is odd. Also δ divides the difference $(y + \sqrt{-26}) - (y - \sqrt{-26}) = 2\sqrt{-26}$, so $\mathrm{N}(\delta)$ divides $4 \cdot 26 = 8 \cdot 13$. Since $\mathrm{N}(\delta)$ is odd, we see that $\mathrm{N}(\delta)$ is 1 or 13. There is no element of $\mathbf{Z}[\sqrt{-26}]$ with norm 13, so $\mathrm{N}(\delta) = 1$. Therefore $\delta = \pm 1$, so $y + \sqrt{-26}$ and $y - \sqrt{-26}$ have only ± 1 as common factors.

If we assume $\mathbf{Z}[\sqrt{-26}]$ has unique factorization, then since $y + \sqrt{-26}$ and $y - \sqrt{-26}$ multiply to a cube and they have only ± 1 as common factors, each of them is a cube. Write

$$y + \sqrt{-26} = (m + n\sqrt{-26})^3,$$

so

$$y = m^3 - 78mn^2 = m(m^2 - 78n^2), \quad 1 = 3m^2n - 26n^3 = n(3m^2 - 26n^2).$$

The second equation tells us $n = \pm 1$. If $n = 1$ then $1 = 3m^2 - 26$, so $3m^2 = 27$, which tells us $m = \pm 3$. Then $y = (\pm 3)(9 - 78) = \pm 207$ and $x^3 = 207^2 + 26 = 42875 = 35^3$, so $x = 35$.

We have discovered new integral solutions to $y^2 = x^3 - 26$, namely $(x, y) = (35, \pm 207)$. If $n = -1$ then $1 = -(3m^2 - 26)$, so $3m^2 = 25$, which has no integral solutions.

Having looked at both possible values of n, we discovered two unexpected integral solutions, but we *missed* the obvious integral solutions $(3, \pm 1)$! How could that happen? The reason is that our argument was based on the assumption of unique factorization in $\mathbf{Z}[\sqrt{-26}]$, but there is not unique factorization in $\mathbf{Z}[\sqrt{-26}]$. A counterexample is

$$27 = 3 \cdot 3 \cdot 3 = (1 + \sqrt{-26})(1 - \sqrt{-26}).$$

It is true that the only integral solutions to $y^2 = x^3 - 26$ are $(3, \pm 1)$ and $(35, \pm 207)$, but a valid proof has to get around the lack of unique factorization in $\mathbf{Z}[\sqrt{-26}]$.

4. Rational solutions

The following table describes all the integral solutions for the cases of Mordell's equation we have looked at. (The examples $k = 1, 2$, and -26 were not fully justified above.)

k	$\mathbf{Z}$-solutions of $y^2 = x^3 + k$
1	$(-1, 0), (0, \pm 1), (2, \pm 3)$
-1	$(1, 0)$
-2	$(3, \pm 5)$
-4	$(2, \pm 2), (5, \pm 11)$
-5	None
6	None
-6	None
7	None
16	$(0, 4), (0, -4)$
-26	$(3, \pm 1), (35, \pm 207)$
45	None
46	None

In each case there are a finite number of integral solutions. If we look instead at rational solutions, we might not get anything new, but we could get a lot more that is new. The next table summarizes the situation.

k	$\mathbf{Q}$-solutions of $y^2 = x^3 + k$
1	$(-1, 0), (0, \pm 1), (2, \pm 3)$
-1	$(1, 0)$
-2	Infinitely many
-4	Infinitely many
-5	None
6	None
-6	None
7	None
16	$(0, 4), (0, -4)$
-26	Infinitely many
45	None
46	Infinitely many

The equations which have more rational solutions than integral solutions are $y^2 = x^3 - 2$, $y^2 = x^3 - 4$, $y^2 = x^3 - 26$, and $y^2 = x^3 + 46$. Examples of non-integral rational solutions to these equations are given in the following table.

k	$\mathbf{Q}$-solution of $y^2 = x^3 + k$
-2	$(129/100, 383/1000)$
-4	$(106/9, 1090/27)$
-26	$(705/4, 18719/8)$
46	$(-7/4, 51/8)$

To emphasize the distinction between classifying integral and rational solutions, we return to $y^2 = x^3 + 16$. We proved the only integral solutions are $(0, \pm 4)$. This does not tell us whether or not there are also rational solutions which are not integral solutions. It turns out there are no further rational solutions, and here is an application. If $a^3 + b^3 = c^3$ with nonzero integers a, b, and c, then the nonzero rational numbers $x = 4bc/a^2$ and $y = 4(a^3 + 2b^3)/a^3$ satisfy $y^2 = x^3 + 16$. (This is taken from [1].) Therefore proving the only *rational* solutions to $y^2 = x^3 + 16$ are $(0, \pm 4)$ forces $x = 0$, but $x = 4bc/a^2 \neq 0$, so this would give a proof of Fermat's Last Theorem for exponent 3.

References

[1] T. R. Bendz, Öfver diophantiska ekvationen $x^n + y^n = z^n$, Ph.D. thesis, Uppsala Univ., 1901.
[2] L. Mordell, "A Chapter in the Theory of Numbers," Cambridge Univ. Press, 1947.