PROOFS BY INDUCTION ON THE NUMBER OF TERMS

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1. Introduction

Mathematical induction is a method that allows us to prove infinitely many similar assertions in a systematic way, by organizing the results in a definite order and showing

- the first assertion is correct ("base case")
- whenever an assertion in the list is correct ("inductive hypothesis"), prove the next assertion in the list is correct ("inductive step").

This tells us every assertion in the list is correct, which is analogous to falling dominos. If dominos are close enough and each domino falling makes the next domino fall, then after the first domino falls all the dominos will fall.

The most basic results that are proved by induction are summation identities, such as

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

for all positive integers n.

Proofs of summation identities are *not* how proofs by induction usually look: we can't always convert a previously settled case *into* the next case by "doing something to both sides." The inductive hypothesis (that is, the assumed truth of previously settled cases) has to help us derive the next case by some method, and that method may or may not start with the inductive hypothesis itself.

Here we will discuss a particular example of identities proved by induction that are not like summation identities: *induction on the number of terms.* Pay attention to the point in the inductive step where the inductive hypothesis is used.

2. Algebra

Theorem 2.1. *For all odd numbers $a_1, \ldots, a_n$ where $n \geq 2$, the product $a_1 \cdots a_n$ is odd.*

Proof. We induct on n, the number of odd numbers.

For the base case $n = 2$ we want to every product of two odd numbers a_1 and a_2 is odd. Since these numbers are odd, $a_1 = 2k_1 + 1$ and $a_2 = 2k_2 + 1$ for some integers k_1 and k_2. Then

$$a_1a_2 = (2k_1+1)(2k_2+1) = 4k_1k_2 + 2k_1 + 2k_2 + 1 = 2(2k_1k_2 + k_1 + k_2) + 1,$$

which is an odd number since $2k_1k_2 + k_1 + k_2 \in \mathbf{Z}$. That settles the base case.

Now assume for an $n \geq 2$ that the result is true for all sets of n odd numbers. To prove the result for $n + 1$, we want to show every set of $n + 1$ odd numbers $a_1, \ldots, a_{n+1}$ has a product $a_1 \cdots a_{n+1}$ that is odd.

A product of $n + 1$ numbers can be written either as a product of 2 numbers or as a product of n numbers by collecting some factors into a single number:

$$a_1a_2\cdots a_{n+1} = (a_1a_2\cdots a_n)a_{n+1} \tag{2.1}$$

is a product of the two numbers $a_1a_2\cdots a_n$ and a_{n+1}, while

$$a_1a_2\cdots a_{n+1} = a_1a_2\cdots a_{n-1}(a_na_{n+1}) \tag{2.2}$$

is a product of the n numbers a_1, $a_2, \ldots, a_{n-1}$, and a_na_{n+1}.

Each of (2.1) and (2.2) leads to a proof of the inductive step: using (2.1) involves the inductive hypothesis (all sets of n odd numbers) and then the base case (all sets of 2 odd numbers) while (2.2) involves the base case (all sets of 2 odd numbers) and then the inductive hypothesis (all sets of n odd numbers).

First method. Write

$$a_1a_2\cdots a_na_{n+1} = (a_1a_2\cdots a_n)a_{n+1}$$

and the product $a_1a_2\cdots a_n$ is odd by the inductive hypothesis (for n odd numbers). Then $(a_1a_2\cdots a_n)a_{n+1}$ is a product of *two* odd numbers, $a_1a_2\cdots a_n$ and a_{n+1}, so their product is odd by the base case. Thus $a_1a_2\cdots a_na_{n+1}$ is an odd number.

Second method. Write

$$a_1a_2\cdots a_na_{n+1} = a_1a_2\cdots a_{n-1}(a_na_{n+1}).$$

The product a_na_{n+1} is an odd number by the base case, so $a_1a_2\cdots a_{n-1}(a_na_{n+1})$ is a product of n odd numbers: a_1, $a_2, \ldots, a_{n-1}$, and a_na_{n+1}. Therefore their product is odd by the inductive hypothesis, which says $a_1a_2\cdots a_{n-1}(a_na_{n+1})$ is odd, so $a_1a_2\cdots a_na_{n+1}$ is odd. □

3. Calculus

Theorem 3.1. *For all sets of differentiable functions $f_1(x), \ldots, f_n(x)$ where $n \geq 2$, the product $f_1(x)\cdots f_n(x)$ is differentiable and*

$$\frac{(f_1(x)\cdots f_n(x))'}{f_1(x)\cdots f_n(x)} = \frac{f_1'(x)}{f_1(x)} + \cdots + \frac{f_n'(x)}{f_n(x)}.$$

Proof. We induct on n, the number of functions.

The base case $n = 2$ follows from the product rule: for any two differentiable functions $f_1(x)$ and $f_2(x)$, the product rule tells us that $f_1(x)f_2(x)$ is differentiable and

$$(f_1(x)f_2(x))' = f_1'(x)f_2(x) + f_1(x)f_2'(x),$$

so dividing both sides by $f_1(x)f_2(x)$ gives us

$$\frac{(f_1(x)f_2(x))'}{f_1(x)f_2(x)} = \frac{f_1'(x)f_2(x) + f_1(x)f_2'(x)}{f_1(x)f_2(x)} = \frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)}.$$

Now assume the result is true for all sets of n differentiable functions where $n \geq 2$. To prove the result for all sets of $n+1$ differentiable functions $f_1(x), \ldots, f_{n+1}(x)$, write their product either as a product of 2 functions or as a product of n functions by collecting some factors into a single function:

$$f_1(x)f_2(x)\cdots f_{n+1}(x) = (f_1(x)f_2(x)\cdots f_n(x)) \cdot f_{n+1}(x) \tag{3.1}$$

is a product of the 2 functions $f_1(x)f_2(x)\cdots f_n(x)$ and $f_{n+1}(x)$, while

$$f_1(x)f_2(x)\cdots f_{n+1}(x) = f_1(x)f_2(x)\cdots f_{n-1}(x)(f_n(x)f_{n+1}(x)) \tag{3.2}$$

is a product of the n functions $f_1(x)$, $f_2(x), \ldots, f_{n-1}(x)$, and $f_n(x)f_{n+1}(x)$.

Each of (3.1) and (3.2) lead to separate proofs of the inductive step: use the inductive hypothesis (all sets of n differentiable functions) and then the base case (all sets of 2 differentiable functions) or use the base case and then the inductive hypothesis.

First method. By the inductive hypothesis, $f_1(x)f_2(x)\cdots f_n(x)$ is differentiable and

$$\frac{(f_1(x)\cdots f_n(x))'}{f_1(x)\cdots f_n(x)} = \frac{f_1'(x)}{f_1(x)} + \cdots + \frac{f_n'(x)}{f_n(x)}. \tag{3.3}$$

Then by the base case of 2 differentiable functions applied to $f_1(x)\cdots f_n(x)$ and $f_{n+1}(x)$, their product $f_1(x)\cdots f_n(x)f_{n+1}(x)$ is differentiable and

$$\begin{aligned} \frac{(f_1(x)\cdots f_n(x)f_{n+1}(x))'}{f_1(x)\cdots f_n(x)f_{n+1}(x)} &= \frac{((f_1(x)\cdots f_n(x))\cdot f_{n+1}(x))'}{(f_1(x)\cdots f_n(x))\cdot f_{n+1}(x)} \\ &= \frac{(f_1(x)\cdots f_n(x))'}{f_1(x)\cdots f_n(x)} + \frac{f_{n+1}'(x)}{f_{n+1}(x)} && \text{by the base case} \\ &= \frac{f_1'(x)}{f_1(x)} + \cdots + \frac{f_n'(x)}{f_n(x)} + \frac{f_{n+1}'(x)}{f_{n+1}(x)}, && \text{by (3.3)} \end{aligned}$$

and this is what we needed to show for $n+1$ differentiable functions.

Second method. By the base case, $f_n(x)f_{n+1}(x)$ is differentiable and

$$\frac{(f_n(x)f_{n+1}(x))'}{f_n(x)f_{n+1}(x)} = \frac{f_n'(x)}{f_n(x)} + \frac{f_{n+1}'(x)}{f_{n+1}(x)}. \tag{3.4}$$

Then by the inductive hypothesis for n differentiable functions applied to $f_1(x), \ldots, f_{n-1}(x)$, and $f_n(x)f_{n+1}(x)$, the product $f_1(x)\cdots f_{n-1}(x)(f_n(x)f_{n+1}(x)) = f_1(x)\cdots f_n(x)f_{n+1}(x)$ is differentiable and

$$\begin{aligned} \frac{(f_1(x)\cdots f_n(x)f_{n+1}(x))'}{f_1(x)\cdots f_n(x)f_{n+1}(x)} &= \frac{(f_1(x)\cdots f_{n-1}(x)(f_n(x)f_{n+1}(x)))'}{f_1(x)\cdots f_{n-1}(x)(f_n(x)f_{n+1}(x))} \\ &= \frac{f_1'(x)}{f_1(x)} + \cdots + \frac{f_{n-1}'(x)}{f_{n-1}(x)} + \frac{(f_n(x)f_{n+1}(x))'}{f_n(x)f_{n+1}(x)} && \text{by ind. hypothesis} \\ &= \frac{f_1'(x)}{f_1(x)} + \cdots + \frac{f_{n-1}'(x)}{f_{n-1}(x)} + \frac{f_n'(x)}{f_n(x)} + \frac{f_{n+1}'(x)}{f_{n+1}(x)} && \text{by (3.4).} \end{aligned}$$

□