MISTAKES IN PRECALCULUS

KEITH CONRAD

In the first section below are valid precalculus formulas using algebra, exponentials, logarithms, and trigonometry. In the second section below are widespread mistakes using algebra, exponentials, logarithms, and trigonometry. If you recognize the mistakes as some that you often make, find a way to relearn the related formulas correctly (*e.g.*, consider special cases of the formula). If you often make a precalculus error that is not listed here, let me know and I may add it.

1. What is Right

Algebra.

$$-(-x) = x, \qquad \frac{1}{1/x} = x, \qquad \frac{1}{a/b} = \frac{b}{a},$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}, \qquad \frac{a}{b}\frac{c}{d} = \frac{ac}{bd}.$$

$$\frac{a+b}{a} = 1 + \frac{b}{a},$$

$$(x+y)^2 = x^2 + 2xy + y^2.$$

When using a calculator, remember the repeating digit patterns

$$.3333\ldots = \frac{1}{3}, \quad .9999\ldots = 1.$$

Similarly, $.19999\ldots = .2$ and $.74999\ldots = .75$.

Exponentials and Logarithms.

Let $b > 0$. The function b^x is defined for all x and takes values in $(0, \infty)$, while the function $\log_b x$ is defined for x in $(0, \infty)$ and takes all possible values. (Note $\log_b x$ is not defined for $x \leq 0$.) The base b exponential and base b logarithm functions are inverses of each other:

$$b^{\log_b x} = x \text{ for all } x > 0 \text{ and } \log_b(b^x) = x \text{ for all } x.$$

In particular, taking $b = e$, so $\log_b x = \ln x$, we have $e^{\ln x} = x$ for all $x > 0$ and $\ln(e^x) = x$ for all x.

Exponential expressions with a single positive base satisfy the rules

$$b^{x+y} = b^x b^y, \quad b^{x-y} = \frac{b^x}{b^y}, \quad (b^x)^y = b^{xy}, \quad b^{-x} = \frac{1}{b^x}.$$

(Example: $b^{2+3} = b^5 = bbbbb = (bb)(bbb) = b^2b^3$, $(b^2)^3 = (b^2)(b^2)(b^2) = (bb)(bb)(bb) = b^6 = b^{2\cdot 3}$.)

Exponential expressions with a base that is a product of two positive numbers satisfy the rules

$$(ab)^x = a^x b^x, \quad \sqrt{ab} = \sqrt{a}\sqrt{b}, \quad \sqrt[3]{ab} = \sqrt[3]{a}\sqrt[3]{b},$$

(Examples: $(ab)^2 = (ab)(ab) = (aa)(bb) = a^2b^2$, $\sqrt{ab} = (ab)^{1/2} = a^{1/2}b^{1/2} = \sqrt{a}\sqrt{b}$, and so on.)

Some logarithmic identities are

$$\log_b(xy) = \log_b x + \log_b y, \quad \log_b\left(\frac{1}{x}\right) = -\log_b x, \quad \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y.$$

Trigonometry.

The fundamental identity of trigonometry is

$$\sin^2 x + \cos^2 x = 1.$$

Remember that $\sin^2 x$ means $(\sin x)^2 = (\sin x)(\sin x)$. Special values that are important to know instinctively are

$$\sin 0 = 0, \quad \cos 0 = 1, \quad \sin\frac{\pi}{2} = 0, \quad \cos\frac{\pi}{2} = 1, \quad \sin\pi = 0, \quad \cos\pi = -1.$$

To understand these equations, look at the point $(\cos x, \sin x)$ on the unit circle when $x = 0, \pi/2$, and π.

From the fundamental identity above, others can be obtained by division. For example, dividing both sides by $\cos^2 x$ gives us

$$\tan^2 x + 1 = \sec^2 x.$$

The addition formulas for sine and cosine, which express the sine and cosine of a sum in terms of sines and cosines of the parts, are useful in explaining some formulas for derivatives in Math 1131. These addition formulas are

$$\sin(x+y) = (\sin x)(\cos y) + (\cos x)(\sin y), \quad \cos(x+y) = (\cos x)(\cos y) - (\sin x)(\sin y).$$

An important special case is $x = y$, called the double-angle formulas:

$$\sin(2x) = 2(\sin x)(\cos x), \quad \cos(2x) = \cos^2 x - \sin^2 x.$$

Using $\cos^2 x = 1 - \sin^2 x$ and $\sin^2 x = 1 - \cos^2 x$, we can rewrite the formula for $\cos(2x)$ using either only $\sin^2 x$ or only $\cos^2 x$:

$$\begin{aligned}\cos(2x) &= (1-\sin^2 x) - \sin^2 x &&= 1 - 2\sin^2 x,\\ \cos(2x) &= \cos^2 x - (1-\cos^2 x) &&= 2\cos^2 x - 1,\end{aligned}$$

so we can write both $\sin^2 x$ and $\cos^2 x$ in terms of $\cos(2x)$:

$$\sin^2 x = \frac{1-\cos(2x)}{2}, \quad \cos^2 x = \frac{1+\cos(2x)}{2}.$$

(In case you remember these two formulas except for uncertainty about the sign in front of $\cos(2x)$ in each one, evaluating them at $x = 0$ tells you how it goes: $\sin^2(0) = 0^2 = 0$ and $\cos^2(0) = 1^2 = 1$, and when $x = 0$ the formula $(1 - \cos(2x))/2$ is $(1-1)/2 = 0$ while $(1 + \cos(2x))/2$ is $(1+1)/2 = 1$. Therefore knowing $\sin 0 = 0$ and $\cos 0 = 1$ tells you that the formula for $\sin^2 x$ in terms of $\cos(2x)$ needs the minus sign and the formula for $\cos^2 x$ in terms of $\cos(2x)$ needs the plus sign.) The expressions for $\sin^2 x$ and $\cos^2 x$ in terms of $\cos(2x)$ are useful for computing certain integrals in Math 1132.

2. What is Wrong

For most values of x and y,

$$(x+y)^2 \neq x^2+y^2, \quad \sqrt{x+y} \neq \sqrt{x}+\sqrt{y}, \quad \frac{1}{x+y} \neq \frac{1}{x}+\frac{1}{y},$$

Putting some of these together, more wrong algebra is

$$\frac{1}{\sqrt{x^2+y^2}} \neq \frac{1}{x}+\frac{1}{y}.$$

$$a^{x+y} \neq a^x + a^y, \quad \log_b(x+y) \neq \log_b x + \log_b y,$$

$$\sin(x+y) \neq \sin x + \sin y, \quad \cos(x+y) \neq \cos x + \cos y.$$

It is almost **never** true that a function $f(x)$ satisfies $f(x+y) = f(x)+f(y)$. The **only** example that fits this is multiplication by a constant: if $f(x) = cx$ then $c(x+y) = cx+cy$. This includes division by c, since $\frac{x}{c} = \frac{1}{c}x$, so $\frac{x+y}{c} = \frac{1}{c}(x+y) = \frac{1}{c}x + \frac{1}{c}y = \frac{x}{c}+\frac{y}{c}$. There are formulas for $(x+y)^2$, for a^{x+y}, and for $\sin(x+y)$ and $\cos(x+y)$ (see the previous section), but there is no standard way to rewrite expressions like $\sqrt{x+y}$, $1/(x+y)$, or $\log_b(x+y)$ in terms of simpler expressions.

Another typical mistake:

$$\frac{a+b}{a} \neq 1+b.$$

When you divide a sum by a, *all* terms must be divided by a. A correct simplifcation for $\frac{a+b}{a}$ is $1+\frac{b}{a}$, which was mentioned on the first page.

Here is another typical algebra error:

$$\frac{1}{1/x+1/y} \neq x+y.$$

Try $x = y = 1$: the left side is 1/2 and the right side is 2. They're not equal. Is there any way to simplify the left side? Yes, if you add the fractions in the denominator first. In two steps,

$$\frac{1}{1/x+1/y} = \frac{1}{(y+x)/xy} = \frac{xy}{x+y}.$$

Do not "solve" for x in a functional relation by treating the function's name as an algebraic symbol:

$$y = \ln x \not\Rightarrow x = \frac{y}{\ln}.$$

Since the functions e^x and $\ln x$ are inverses of each other, the correct statement is that $y = \ln x \Rightarrow x = e^y$ by raising e to both sides of $y = \ln x$ to undo the logarithm on the right side ($e^{\ln x} = x$).

Finally, remember that a calculator's values are often just (good) approximations, so you need to exercise judgment to properly understand what it is telling you. For example, if your answer to a multiple-choice problem is (9/5)(1/3) and the choices are (a) 1/3, (b) 1/5, (c) 3/5, and (d) None of these, the right answer is (c) because (9/5)(1/3) *simplifies* to 3/5 by algebra with fractions. If you instead use a calculator, it may report the value of (9/5)(1/3) as .5999999 and you need to realize that means $.5999\ldots = .6 = 6/10 = 3/5$, which is (c).

3. General Advice

If you are uncertain about whether an algebraic identity is true, set the variables equal to 1 and see if it is true. This will often serve to reveal that a false identity really is false. If the result you get when all variables equal 1 looks correct, try another choice for the variables as a backup check, like letting some be 2 and others be 3 (don't let everything be 2 or you may be misled because $2 + 2$ equals $2 \cdot 2$).

Example. Is $(x + y)^2 = x^2 + y^2$? When $x = y = 1$ this says $2^2 = 2$, which is wrong.

Example. Is $\sqrt{x + y} = \sqrt{x} + \sqrt{y}$? When $x = y = 1$ this says $\sqrt{2} = 2$, which is wrong.

Example. Is $\sin(xy) = (\sin x)(\sin y)$? When $x = y = 0$ this says $0 = 0$, but try $x = y = \pi$ or $x = y = \pi/2$.

Have a healthy skepticism about algebraic identities that by experience you often misremember (*e.g.*, if you make errors with exponentials).

See http://www.math.vanderbilt.edu/~schectex/commerrs/ for many common errors in math.