AN EXAMPLE OF PARTIAL FRACTIONS

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We describe here how to calculate

$$\int \frac{x+7}{x^3+9x}\,dx$$

by first getting the partial fraction decomposition of $(x+7)/(x^3+9x)$ in two ways.

The decomposition takes the form

$$\frac{x+7}{x^3+9x} = \frac{x+7}{x(x^2+9)} = \frac{A}{x} + \frac{B+C}{x^2+9}$$

since the quadratic factor x^2+9 in the denominator does not factor further. Clearing the denominator, we have

$$x+7 = A(x^2+9) + (Bx+C)x. \tag{1}$$

We present two ways to find A, B, and C from (1).

Method 1. Setting $x = 0$ in (1) kills off the second term, and we get

$$7 = A(9) + 0 = 9A \Longrightarrow A = \frac{7}{9},$$

so now we know A. Unfortunately there is not a number we can plug in for x in (1) to isolate just B or just C. Instead let's plug in two simple choices, say 1 and -1 (since we haven't use them yet; if we had then we could try 2 and -2, or 2 and 3, or $\ldots$). At $x = 1$ and at $x = -1$, (1) becomes

$$\begin{aligned} 8 &= 10A + B + C = \frac{70}{9} + B + C, \\ 6 &= 10A + (-B+C)(-1) = \frac{70}{9} + B - C. \end{aligned}$$

Adding the two equations cancels C:

$$14 = \frac{140}{9} + 2B \Longrightarrow 7 = \frac{70}{9} + B \Longrightarrow B = 7 - \frac{70}{9} = \frac{63}{9} - \frac{70}{9} = -\frac{7}{9}.$$

Now we know B, so we can solve for C as

$$C = 8 - \frac{70}{9} - B = \frac{72}{9} - \frac{70}{9} + \frac{7}{9} = \frac{72-70+7}{9} = \frac{9}{9} = 1.$$

Method 2. Collect like powers of x together on both sides of (1):

$$x + 7 = (A+B)x^2 + Cx + 9A.$$

Coefficients of like powers of x on both sides must agree, so

$$A + B = 0, \quad C = 1, \quad 9A = 7.$$

The last equation tells us $A = 7/9$ and the first equation then tells us $B = -A = -7/9$.

Now that we have (by either method) the partial fraction decomposition

$$\frac{x+7}{x(x^2+9)} = \frac{7/9}{x} + \frac{-(7/9)x+1}{x^2+9}$$

we can integrate the left side by integrating the right side:

$$\int \frac{x+7}{x(x^2+9)}\,dx = \int \frac{7/9}{x}\,dx + \int \frac{-(7/9)x+1}{x^2+9}\,dx = \frac{7}{9}\int \frac{1}{x}\,dx + \int \frac{-(7/9)x+1}{x^2+9}\,dx. \tag{2}$$

On the right, an antiderivative of $1/x$ is $\ln|x|$. The second integral can be split up:

$$\int \frac{-(7/9)x+1}{x^2+9}\,dx = \int \frac{-(7/9)x}{x^2+9}\,dx + \int \frac{1}{x^2+9}\,dx = -\frac{7}{9}\int \frac{x}{x^2+9}\,dx + \int \frac{1}{x^2+9}\,dx.$$

The first integral on the right, using the substitution $u = x^2+9$ (so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}du$) becomes

$$\int \frac{x}{x^2+9}\,dx = \int \frac{(1/2)}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+9) + C,$$

where we can drop the absolute value signs because x^2+9 is *always positive*. An antiderivative of $1/(x^2+9)$ is $\frac{1}{3}\arctan(x/3)$ (use the substitution $x = 3t$ to write the integral in terms of t if you want to carry out the details of that), so

$$-\frac{7}{9}\int \frac{x}{x^2+9}\,dx + \int \frac{1}{x^2+9}\,dx = -\frac{7}{18}\ln(x^2+9) + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C.$$

Feeding this into (2), we finally get

$$\int \frac{x+7}{x(x^2+9)}\,dx = \frac{7}{9}\ln|x| - \frac{7}{18}\ln(x^2+9) + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C.$$

As a check on your understanding, use Method 1 and Method 2 above to derive the partial fraction decomposition

$$\frac{x^2+3x+1}{x(x^2+25)} = \frac{1/25}{x} + \frac{(24/25)x+3}{x^2+25}$$

and then use that to show

$$\int \frac{x^2+3x+1}{x(x^2+25)}\,dx = \frac{1}{25}\ln|x| + \frac{12}{25}\ln(x^2+25) + \frac{3}{5}\arctan\left(\frac{x}{5}\right) + C.$$