ESCAPE VELOCITY: AN APPLICATION OF IMPROPER INTEGRALS

KEITH CONRAD

The "escape velocity" from Earth is the minimum velocity needed in order to leave Earth's gravitational field. This is the speed needed by rockets going outside of Earth orbit (to the moon, to Mars, and so on). Computing this velocity is a nice use of improper integrals. To follow the calculation you'll need to know about kinetic energy ($\frac{1}{2}mv^2$), which is the energy associated to motion, and the concept of work in physics.

To find the escape velocity, we ask: how much work is needed to move an object, against the force of gravity alone, infinitely far from the Earth? Surprisingly, it turns out only a *finite* amount of work is needed! This is based on Newton's law of gravitation, which says the gravitational force between two objects of mass m_1 and m_2 and distance r is

$$F = G\frac{m_1 m_2}{r^2},$$

where G is a universal constant, called the gravitational constant. It is approximately $6.673 \times 10^{-11} \frac{\text{m}^3}{\text{kg}\cdot\text{sec}^2}$. The gravitational force due to Earth is what we must act against when we want to move an object from the planet's surface out to infinity.

The gravitational force between a planet and an object above it is $F(r) = GMm/r^2$, where M is the planet's mass, m is the object's mass, and r is the distance from the object to the *center* of the planet. (This is based on the assumption that the planet is spherical.) The *work* needed to move an object against the force of Earth's gravity starting at a distance r from the center of Earth out to a distance $r + dr$ is $F(r)\,dr = (GMm/r^2)dr$. Letting R be the radius of the Earth, the total work needed to move an object an infinite distance from the Earth's surface is the integrated force of gravity times the infinitesimal distance dr from $r = R$ out to $r = \infty$:

$$W = \int_R^\infty F(r)\,dr = \int_R^\infty G\frac{Mm}{r^2}\,dr.$$

This is an improper integral, and its value is finite because $\int_R^\infty dr/r^2$ converges:

$$W = GMm\int_R^\infty \frac{dr}{r^2} = GMm\left(-\frac{1}{r}\bigg|_R^\infty\right) = \frac{GMm}{R}.$$

Blasting the object off the Earth at an initial velocity v_0 gives it initial kinetic energy $\frac{1}{2}mv_0^2$ (recall m denotes the object's mass). The escape velocity is the velocity that would convert all the kinetic energy to the work W. Equate W and initial kinetic energy:

$$\frac{GMm}{R} = \frac{1}{2}mv_0^2 \Longrightarrow v_0^2 = \frac{2GM}{R} \Longrightarrow v_0 = \sqrt{\frac{2GM}{R}}.$$

Feeding into this the estimates $G \approx 6.673 \times 10^{-11} \frac{\text{m}^3}{\text{kg}\cdot\text{sec}^2}$, $M \approx 5.9736 \times 10^{24}$ kg, and $R \approx 6.4 \times 10^6$ m, we get $v_0 \approx 11161$ m/sec, or roughly 11.2 km/sec, which is about 7 miles per second. This is very fast: it's over 30 times the speed of sound. (The fastest spacecraft ever, according to this page, is the Parker Solar Probe, which in 2024 will be traveling around the Sun at 120 miles per second or 192 km per second.)

An interesting aspect of this calculation is how m, the object's own mass, canceled when we set the work equal to the kinetic energy: the escape velocity v_0 is *independent* of the mass of the object being launched away from Earth.

Remark. For a spacecraft to reach Earth orbit, the necessary *orbital velocity* is less than the escape velocity since we're not trying to leave the influence of the Earth's gravity completely. It turns out orbital velocity is smaller than escape velocity by a factor of $\sqrt{2}$: it is $\sqrt{GM/R} = \sqrt{2GM/R}/\sqrt{2} = v_0/\sqrt{2} \approx 7.9$ km/sec, or 4.9 miles/sec.

Knowing the escape velocity, we can work out how much of a spaceflight's initial mass can be taken up by anything other than fuel (such as astronauts, satellites, the rocket structure itself, *etc.*) if the rocket is going to escape the Earth's gravitational pull. If a rocket with

- initial mass m_0,
- final mass m_f (the mass after the fuel is used up, *not* "mass of fuel"),
- and exhaust velocity $v_{\rm ex}$ (the speed of exhaust pushed out the back of the rocket)

achieves a final velocity v_f then the rocket equation says

$$v_f = v_{\rm ex} \ln\left(\frac{m_0}{m_f}\right).$$

Solving this for the mass ratio,

$$\frac{m_0}{m_f} = e^{v_f/v_{\rm ex}}.$$

A typical value for the exhaust velocity $v_{\rm ex}$ is 4 km/sec. Setting v_f to be the escape velocity 11.2 km/sec, we get

$$\frac{m_0}{m_f} \approx e^{11.2/4} \approx 16.4,$$

so the proportion of rocket mass at launch that is *not* fuel is

$$\frac{m_f}{m_0} = \frac{1}{16.4} \approx .06 = 6\%.$$

To escape Earth's graviational pull we need $v_f \geq 11.2$ km/sec, so $m_0/m_f \geq 16.4$. Thus $m_f/m_0 \leq 6\%$: *over 90% of the rocket mass at launch is the fuel.*

If the exhaust velocity goes down, then the rocket has less thrust and it makes sense that even more fuel is needed in the rocket to achieve escape velocity. For example, if $v_{\rm ex}$ changes from 4 km/sec to 3 km/sec then $m_0/m_f = e^{11.2/3} \approx 41.8$ so $m_f/m_0 \approx 1/41.8 \approx .024 = 2.4\%$: less than 3% of the initial rocket mass is now available for things other than fuel.

If the mass of the earth doubled, keeping the same radius, then the escape velocity on Earth would change from 11.2 km/sec to $\sqrt{2G(2M)/R} \approx \sqrt{2}(11.2) \approx 15.8$ km/sec. A rocket with the same exhaust velocity as initially mentioned, 4 km/sec, would now need $m_0/m_f = e^{15.8/4} \approx 52$ (over three times the previous value of 16.4) to reach escape velocity, making the proportion of rocket mass at launch that's not fuel equal to $1/52 \approx 1.9\%$. Previously that proportion was 6%. Since $6/1.9 \approx 3.15$, the proportion of rocket mass at launch that's not fuel has shrunk by over a factor of 3. If the mass of the earth increased by a factor of 10, keeping the same radius, then the escape velocity from Earth would be $\sqrt{10}(11.2) \approx 35.4$ km/sec, over a 3-fold increase of the actual value 11.2 km/sec, and a rocket with exhaust velocity 4 km/sec would reach escape velocity only if $m_0/m_f = e^{35.4/4} \approx 6974$, so $m_f/m_0 \approx .0001$: that's 1/100-th of 1%. Such a rocket is essentially all fuel at launch, making meaningful space travel totally impractical.

The lesson here is that even though we say human civilization now is in the "space age", we are lucky to be able to launch anything into orbit at all.