APPROXIMATING π WITH INTEGRATION

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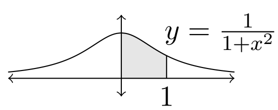
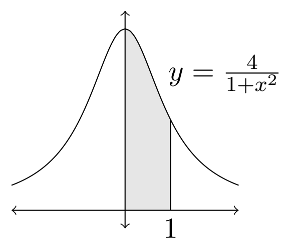
One of the advances in mathematics that came out of calculus is that it led to methods of getting very good approximations to π. We will see the idea behind this by representing π as a definite integral and then using error estimates in numerical integration.

First we show how to write π as a definite integral. Since $(\arctan x)' = \dfrac{1}{1+x^2}$, we get

$$\int_0^1 \frac{dx}{1+x^2} = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4} \Longrightarrow \boxed{\pi = \int_0^1 \frac{4}{1+x^2}\,dx.}$$

This says π is the area of the shaded region in the second picture below.





Using Simpson's rule on this integral, $S_4 = 3.141568\ldots$ and $S_6 = 3.141591\ldots$. To be certain that S_n gives us π to a desired accuracy, we use the error bound for Simpson's rule, which says

$$\left|\int_0^1 \frac{4}{1+x^2}\,dx - S_n\right| \leq \frac{K}{180}\frac{(b-a)^5}{n^4} = \frac{K}{180n^4},$$

where K is an upper bound on the fourth derivative of $\frac{4}{1+x^2}$ over $[0,1]$. Letting $f(x) = \frac{4}{1+x^2}$, it turns out that $f^{(4)}(x) = \frac{96(1-10x^2+5x^4)}{(1+x^2)^5}$ and $|f^{(4)}(x)| \leq 96$ on $[0,1]$. Therefore we can use $K = 96$, so

$$\left|\int_0^1 \frac{4}{1+x^2}\,dx - S_n\right| \leq \frac{96}{180n^4}.$$

Example. $|\pi - S_n| < 1/10^4$ if

$$\frac{96}{180n^4} < \frac{1}{10^4} \Longleftrightarrow n^4 > \frac{96 \cdot 10^4}{180} \approx 5333.3 \Longleftrightarrow n \geq 9.$$

Take $n = 10$ since n must be even for Simpson's rule: $S_{10} = 3.1415926\ldots$ so

$$\boxed{S_{10} - .0001 \approx \underline{3.141}49 \leq \pi \leq S_{10} + .0001 \approx \underline{3.141}69.}$$

Example. $|\pi - S_n| < 1/10^7$ if

$$\frac{96}{180n^4} < \frac{1}{10^7} \Longleftrightarrow n^4 > \frac{96 \cdot 10^7}{180} = 5333333.3 \Longleftrightarrow n \geq 49.$$

Since n is even in Simpson's rule we take $n \geq 50$: $S_{50} = 3.141592653587\ldots$, so

$$\boxed{S_{50} - .0000001 \approx \underline{3.141592}5 \leq \pi \leq S_{50} + .0000001 \approx \underline{3.141592}7.}$$