VOLUME WITH UNIT SQUARE BASE AND HEIGHT 4

We take horizontal slices from the base square up to the point at the top. Let's call the vertical direction y, so $0 \leq y \leq 4$. The y-slice is a square, and what is its area $A(y)$? The area of any square with side length s is s^2, so we need to figure out the side length $s(y)$ of the square slice at height y. Then $A(y) = s(y)^2$ and the volume will be

$$\int_0^4 A(y)\,dy = \int_0^4 s(y)^2\,dy.$$

We will use a similar triangle argument to find $s(y)$. If you cut through the total solid *vertically* (not horizontally like the slices being made), parallel to two sides of the base and through the middle, you'll get a tall triangle with base 1 and height 4. Within this triangle there is a smaller similar triangle with the same top part and its base is the y-slice with length $s(y)$. What is the height of this smaller triangle? Its base is y units above the bottom, and the total height of the solid is 4, so this smaller triangle has height $4 - y$.

Thus we have two similar triangles, one (big) triangle with base 1 and height 4 and another (smaller) triangle tucked into its top with base $s(y)$ and height $4 - y$. Since they are similar,

$$\frac{\text{base}}{\text{height}} = \frac{1}{4} = \frac{s(y)}{4-y} \Longrightarrow s(y) = \frac{4-y}{4} = 1 - \frac{y}{4}.$$

As a check on our work, this formula says $s(0) = 1$ and $s(4) = 0$, which makes geometric sense: when $y = 0$ we're at the bottom, where the y-slice has side length 1, and when $y = 4$ we're at the top (a point) where the y-slice is a point with "side length" 0.

Now we feed this formula for the side length of the y-slice into the volume integral:

$$\int_0^4 A(y)\,dy = \int_0^4 s(y)^2\,dy = \int_0^4 \left(1 - \frac{y}{4}\right)^2\,dy.$$

This is a polynomial integral which you can check is $\frac{4}{3}$.