# VOLUME WITH UNIT EQUILATERAL TRIANGLE BASE AND SQUARE SLICES

The equilateral triangle at the bottom is given to us with sides of length 1. We form a solid from square cross-sections over this triangle, running from one vertex to the opposite side. Let's label these square slices using a variable $x$ which is 0 at the vertex and is – what? – at the opposite side? We need to know the height of this bottom triangle to figure out how far the vertex is from the opposite side. An equilateral triangle with side of length $s$ and height $h$ have these parameters related by $h = \frac{\sqrt{3}}{2}s$ (this is trigonometry), so the height of an equilateral triangle with side length 1 is $\frac{\sqrt{3}}{2} \cdot 1 = \frac{\sqrt{3}}{2}$. Thus $0 \leq x \leq \frac{\sqrt{3}}{2}$. Numerically, $\frac{\sqrt{3}}{2} \approx .866$, which is why in the film you see one side hitting a line somewhere between .5 and 1: it's touching at nearly .866.

The $x$-slice through this solid is a square whose area $A(x)$ is $s(x)^2$, where $s(x)$ is the side length of that $x$-slice. We can see this side length in the bottom triangle: it's the length of the cut through the base triangle made by the $x$-slice. Notice that this line segment at the $x$-slice in the bottom triangle is the side of an equilateral triangle with height $x$ tucked into the whole bottom triangle. Since its height is $x$, its side length is

$$s(x) = \frac{2}{\sqrt{3}}x.$$

Therefore

$$A(x) = s(x)^2 = \frac{4}{3}x^2,$$

so the volume of the solid is

$$\int_0^{\sqrt{3}/2} A(x)\,dx = \int_0^{\sqrt{3}/2} \frac{4}{3}x^2\,dx = \frac{4}{3}\int_0^{\sqrt{3}/2} x^2\,dx = \frac{4}{3} \cdot \frac{(\sqrt{3}/2)^3}{3} = \frac{\sqrt{3}}{6}.$$