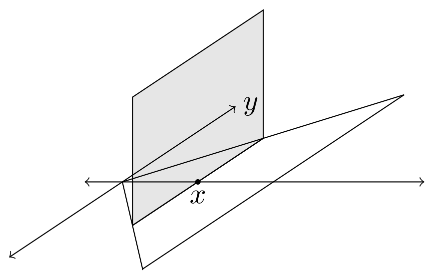
VOLUME WITH UNIT EQUILATERAL TRIANGLE BASE AND SQUARE SLICES

The equilateral triangle at the base has sides of length 1. Form a solid using square cross-sections laid out over this triangle, running from one vertex (the origin of coordinates in the picture below) to the opposite side. Label these square slices using a variable x which is 0 at the vertex and is some positive value at the opposite side.



To find the height of the bottom triangle we figure out how far the origin is from the opposite side. For an equilateral triangle with side of length s and height h, these parameters are related by $h = \frac{\sqrt{3}}{2}s$ (from trigonometry), so the height of an equilateral triangle with side length 1 is $\frac{\sqrt{3}}{2} \cdot 1 = \frac{\sqrt{3}}{2}$. Thus the base triangle has $0 \leq x \leq \frac{\sqrt{3}}{2} \approx .866$.

The x-slice is a square whose area $A(x)$ is $s(x)^2$, where $s(x)$ is the side length of the x-slice. This side length in the bottom triangle is the length of the cut through the bottom triangle made by the x-slice, which is the side of an equilateral triangle tucked inside the bottom triangle with height x, so its side length is

$$s(x) = \frac{2}{\sqrt{3}}x.$$

Therefore

$$A(x) = s(x)^2 = \frac{4}{3}x^2,$$

so the volume of the solid is

$$\int_0^{\sqrt{3}/2} A(x)\,dx = \int_0^{\sqrt{3}/2} \frac{4}{3}x^2\,dx = \frac{4}{3}\int_0^{\sqrt{3}/2} x^2\,dx = \frac{4}{3} \cdot \frac{(\sqrt{3}/2)^3}{3} = \frac{\sqrt{3}}{6}.$$