DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

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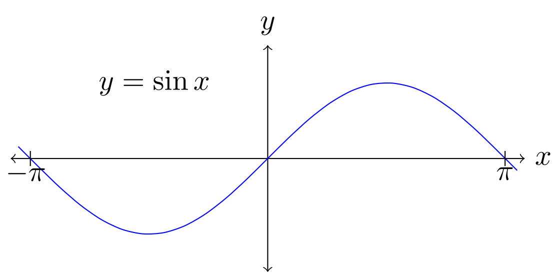
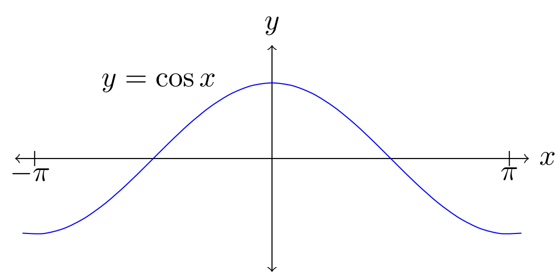
There are six basic trigonometric functions:

$$\sin x, \quad \cos x, \quad \tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

We will always regard the angle x as being in *radians*. To compute the derivatives of these functions, we start with $\sin x$ and $\cos x$. The derivatives of the other trigonometric functions will follow from these two using the quotient rule.

Below are the graphs of $\sin x$ and $\cos x$.





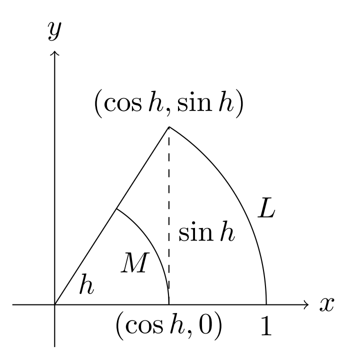
First we find the derivatives of $\sin x$ and $\cos x$ at $x = 0$:

$$\sin'(0) = \lim_{h\to 0} \frac{\sin h}{h}, \quad \cos'(0) = \lim_{h\to 0} \frac{\cos h - 1}{h}. \tag{1}$$

The graph of $\cos x$ has a horizontal tangent line at $x = 0$, so $\cos'(0) = 0$. Determining $\sin'(0)$ is more subtle. From the graph of $\sin x$ we can see that $\sin'(0) > 0$. Using *radian measure* for angles we will show $\sin'(0)$ in fact is 1.

Claim. $\lim_{h\to 0} \frac{\sin h}{h} = 1$.

Proof of Claim: Pick a small $h > 0$. In the figure below we mark off an angle h with respect to the positive x-axis. Let L be the length of a circular arc having radius 1 and angle h, and M be the length of a circular arc starting at $(\cos h, 0)$ with angle h.



We will make a comparison of arc lengths. The circular arc with length L is clearly longer than the vertical segment with length $\sin h$, and this vertical segment in turn is longer than the circular arc with with length M: $M < \sin h < L$. What are L and M?

When an angle θ is measured *in radians*, the length of a circular arc it cuts out along a circle of radius r is $r\theta$. The arc with length L is on a circle with radius 1 and the arc with length M is on a circle with radius $\cos h$, so *if we measure h in radians* we get $L = 1 \cdot h = h$ and $M = (\cos h)h$. Thus the inequality $M < \sin h < L$ we found above says $(\cos h)h < \sin h < h$, and if we divide by h we get

$$\cos h < \frac{\sin h}{h} < 1. \tag{2}$$

This was derived for small $h > 0$. Since $\cos h$ and $(\sin h)/h$ are even functions, the inequality (2) is true for small $h < 0$ too, and hence it's true for all small $h \neq 0$.

As $h \to 0$ we have $\cos h \to \cos 0 = 1$ by continuity. Thus (2) tells us $(\sin h)/h \to 1$ as $h \to 0$ by the squeeze theorem. This proves $\sin'(0) = 1$ *when angles are in radians.*

To find the derivatives of $\sin x$ and $\cos x$ for general x we use their addition formulas:

$$\sin(a+b) = \sin a \cos b + \cos a \sin b, \quad \cos(a+b) = \cos a \cos b - \sin a \sin b.$$

The ratio in the limit definition of the derivative of $\sin x$ is

$$\begin{aligned} \frac{\sin(x+h) - \sin x}{h} &= \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\ &= \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}. \end{aligned}$$

Letting $h \to 0$ and looking back at (1) tells us

$$\begin{aligned} \sin'(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h} \\ &= (\sin x)(\cos'(0)) + (\cos x)(\sin'(0)) \\ &= (\sin x) \cdot 0 + (\cos x) \cdot 1 \\ &= \cos x. \end{aligned}$$

Turning to the derivative of $\cos x$,

$$\begin{aligned} \cos'(x) &= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\ &= \lim_{h \to 0} \cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \\ &= \cos x \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h} \\ &= (\cos x)(\cos'(0)) - (\sin x)(\sin'(0)) \\ &= -\sin x. \end{aligned}$$

Using the quotient rule we get the derivative of $\tan x = \sin x / \cos x$:

$$\tan' x = \frac{\sin' x \cos x - \cos' x \sin x}{\cos^2 x} = \frac{\cos x \cos x - (-\sin x) \sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

Since $\sin^2 x + \cos^2 x = 1$ the formula simplifies to $\tan' x = 1/\cos^2 x = \sec^2 x$. The derivatives of the other trigonometric functions can be found in a similar way:

$$\cot' x = \left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos' x)(\sin x) - (\cos x)\sin' x}{\sin^2 x} = \frac{-\cos^2 x - \sin^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

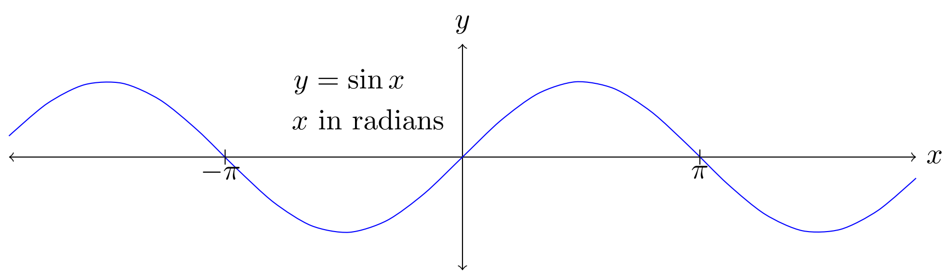
and

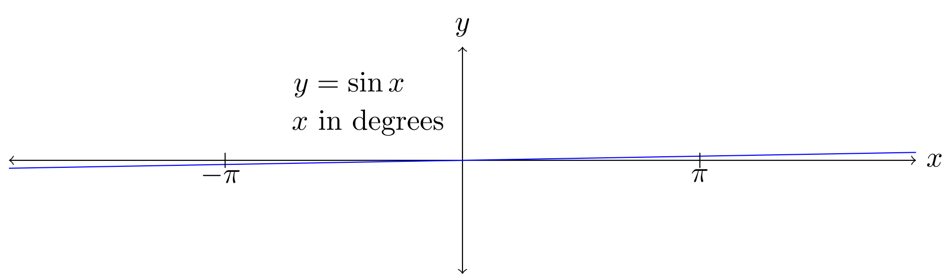
$$\sec' x = \left(\frac{1}{\cos x}\right)' = \frac{-\cos' x}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

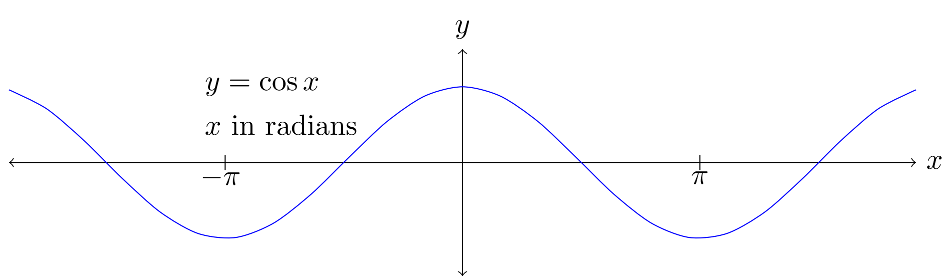
and

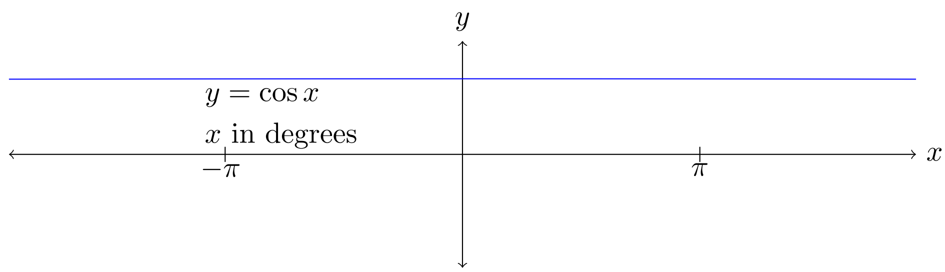
$$\csc' x = \left(\frac{1}{\sin x}\right)' = \frac{-\sin' x}{\sin^2 x} = -\frac{\cos x}{\sin^2 x} = -\csc x \cot x.$$

Remark. It is critical that we measure angles in radians for the above derivative formulas to be correct. If we measure angles in degrees then the formulas are wrong. Compare the graphs of $\sin x$ and $\cos x$ below, all drawn to the same scale, when x is measured first in radians and then in degrees. For example, when we measure angles in degrees the number $\sin'(0)$ is nearly 0 Precisely, when angles are in degrees $\sin'(0) = \pi/180 \approx .0174$.








y
$y = \cos x$
x in degrees
x
$-\pi$
π