REVIEW OF LOGARITHMS

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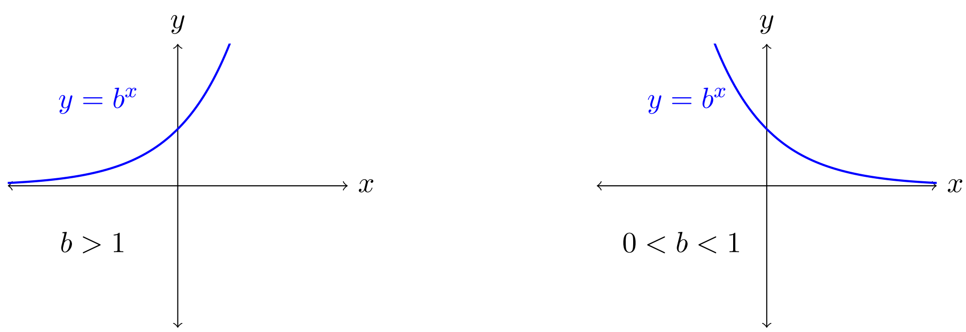
For a number $b > 0$ with $b \neq 1$, the function b^x has a graph that looks like one of those below, depending on whether $b > 1$ (left) or $0 < b < 1$ (right).



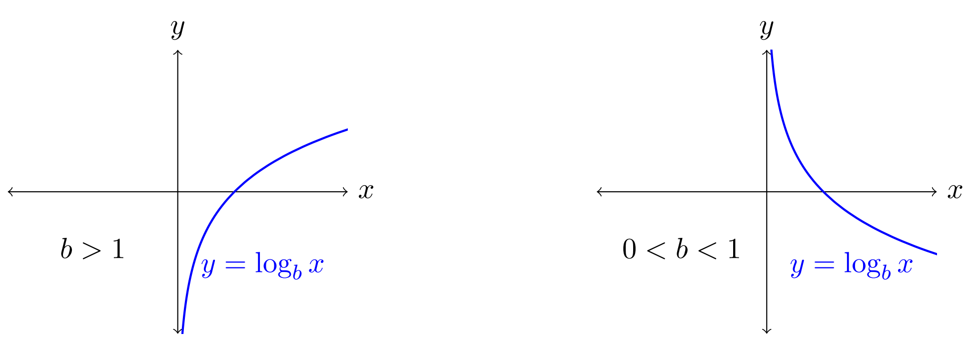
The function b^x for $b > 0$ with $b \neq 1$ has domain the interval $(-\infty, \infty)$ of all real numbers and range the interval $(0, \infty)$ of all positive numbers. This function is either increasing (if $b > 1$) or decreasing (if $0 < b < 1$). In both cases the function b^x is one-to-one:

$$x_1 \neq x_2 \Longrightarrow b^{x_1} \neq b^{x_2}.$$

Therefore the function b^x has an inverse with domain $(0, \infty)$ and range $(-\infty, \infty)$: the inverse function at a positive number x is the number y fitting $b^y = x$. We call y the base-b logarithm of x, written as $\log_b x$, so $\log_b x$ is *the one number for which b raised to that power is x*. That is, $b^{\log_b x} = x$, and also $\log_b(b^x) = x$.

Example. Since $4 = 2^2$ and $8 = 2^3$, $\log_2 4 = 2$ and $\log_2 8 = 3$. Since $2^1 < 3 < 2^2$, $\log_2 3$ lies between the exponents 1 and 2. More precisely, $\log_2 3 = 1.5849\ldots$: this solves $2^y = 3$.

A graph of $y = \log_b x$ is formed by flipping the graph of $y = b^x$ across the line $y = x$, and is illustrated below. It has the y-axis as a vertical asymptote and no other asymptotes.



Exponential functions satisfy several basic identities:

$$b^u b^v = b^{u+v}, \qquad \frac{b^u}{b^v} = b^{u-v}, \qquad (b^u)^v = b^{uv}.$$

Here are corresponding formulas for logarithms:

$$\log_b(xy) = \log_b x + \log_b y \text{ for } x, y > 0, \tag{1}$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y \text{ for } x, y > 0, \tag{2}$$

$$\log_b(x^y) = y \log_b x \text{ for } x > 0. \tag{3}$$

To derive each of the formulas in (1)–(3) we rely on the characteristic property of a logarithm value: $\log_b x$ is the only number y satisfying the equation $b^y = x$.

Proof of (1): Let $u = \log_b x$ and $v = \log_b y$, so $b^u = x$ and $b^v = y$. Thus $xy = b^u b^v = b^{u+v}$, so $\log_b(xy) = u + v = \log_b x + \log_b y$.

Proof of (2): Let $u = \log_b x$ and $v = \log_b y$, so $b^u = x$ and $b^v = y$. Thus $x/y = b^u/b^v = b^{u-v}$, so $\log_b(x/y) = u - v = \log_b x - \log_b y$.

Proof of (3): Let $u = \log_b(x^y)$, so $b^u = x^y$. Also $x = b^{\log_b x}$, so $x^y = (b^{\log_b x})^y = b^{(\log_b x)y} = b^{y \log_b x}$. Since $b^u = b^{y \log_b x}$ we get $u = y \log_b x$, so $\log_b(x^y) = y \log_b x$.

Example. Using formula (3), $\log_2(\sqrt{3}) = \log_2(3^{1/2}) = \frac{1}{2}\log_2 3$ and $\log_5(1/3^2) = \log_5(3^{-2}) = -2\log_5 3$.

Example. While $x^2 > 0$ for $x \neq 0$, the formula $\log_b(x^2) = 2\log_b x$ is only true for $x > 0$. The correct formula when $x \neq 0$ is $\log_b(x^2) = 2\log_b |x|$ since $x^2 = |x|^2$ and $|x| > 0$.

Warning. Avoid bogus algebraic identities. While there are formulas for logarithms of multiplicative expressions like xy, x/y, and x^y, there is no formula for logarithms of additive expressions: $\log_b(x+y)$ or $\log_b(x-y)$ can't be written in terms of $\log_b x$ and $\log_b y$. Unless you write the number inside a logarithm as a product, ratio, or power, *there is no identity for it*.

The logarithm formulas above all involve a single base. There is an additional formula for logarithms involving two bases b and c, called the change of base formula:

$$\log_b a = \frac{\log_c a}{\log_c b}.$$

To derive this formula, rewrite it as a product: $(\log_b a)(\log_c b) \stackrel{?}{=} \log_c a$. We will show this equation is true by raising c to the left side:

$$c^{(\log_b a)(\log_c b)} = (c^{\log_c b})^{\log_b a} = b^{\log_b a} = a.$$

The only solution to $c^x = a$ is $\log_c a$, so $(\log_b a)(\log_c b) = \log_c a$.

The change of base formula lets us write a logarithm function in any base b in terms of a logarithm function in any other base c:

$$\log_b x = \frac{\log_c x}{\log_c b} = \frac{1}{\log_c b}\log_c x.$$

This means that up to a scaling factor *there is basically only one logarithm function*! For example, base 2 logarithms can be written in terms of base 10 logarithms and in terms of (base e) natural logarithms:

$$\log_2 x = \frac{\log_{10} x}{\log_{10} 2} = \frac{\ln x}{\ln 2}.$$

This kind of formula is important to be aware of if you want to calculate a logarithm to base 2 on a calculator and you only have buttons for $\log_{10}$ and ln.

While base 10 logarithms are the main kind of logarithm seen in high school, in math and physics the preferred base for logarithms is e on account of special properties of natural logarithms in calculus. In computer science, due to the use of binary representations, the preferred base for logarithms is often 2.