LINEARIZATION AND DIFFERENTIALS

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For a function $y(x)$ that is differentiable at a number a, the function

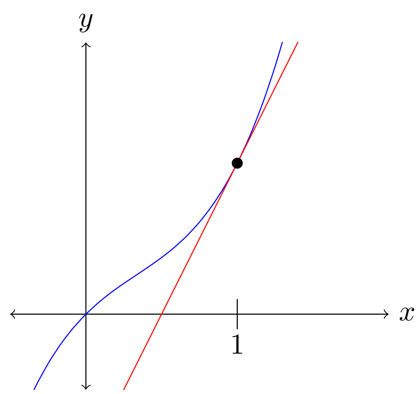
$$L(x) = y(a) + y'(a)(x - a)$$

is called the *linearization of* $y(x)$ *at* a. This is the linear function whose graph is the tangent line to the graph of $y(x)$ at $x = a$. Here are several examples of linearizations, with the graph of $y(x)$ in blue and the graph of $L(x)$ in red.

Example 1. Linearize $x^3 - x^2 + x$ at 1.

Here $y(x) = x^3 - x^2 + x$ and $a = 1$. Since $y'(x) = 3x^2 - 2x + 1$, the linearization of $y(x)$ at 1 is

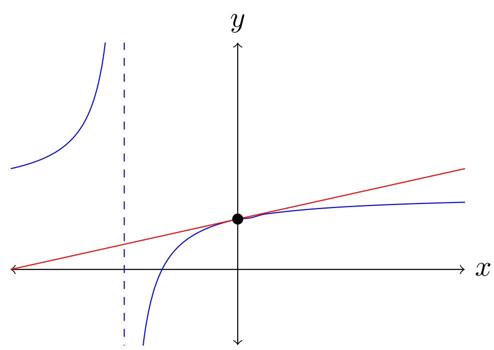
$$L(x) = y(1) + y'(1)(x - 1) = 1 + 2(x - 1) = 2x - 1.$$



Example 2. Linearize $\dfrac{1 + 2x}{3 + 4x}$ at 0.

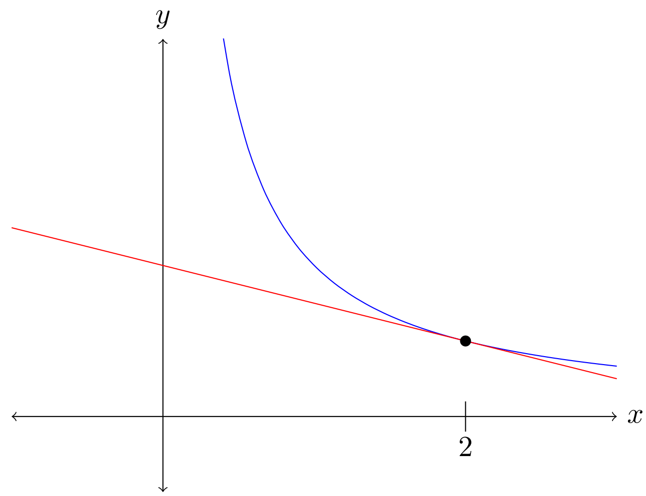
Here $y(x) = (1 + 2x)/(3 + 4x)$ and $a = 0$. Using $y'(x) = 2/(3 + 4x)^2$, the linearization of $y(x)$ at 0 is

$$L(x) = y(0) + y'(0)(x - 0) = \frac{1}{3} + \frac{2}{9}x.$$



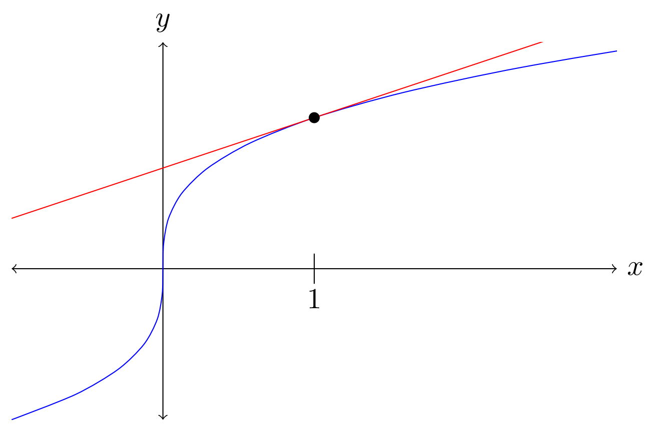
Example 3. Linearize $1/x$ at 2.
Here $y(x) = 1/x$, so $y'(x) = -1/x^2$ and the linearization of $y(x)$ at 2 is

$$L(x) = y(2) + y'(2)(x-2) = \frac{1}{2} - \frac{1}{4}(x-2) = -\frac{1}{4}x + 1.$$



Example 4. Linearize $\sqrt[3]{x}$ at 1.
Here $y(x) = \sqrt[3]{x}$, so $y'(x) = \frac{1}{3}x^{-2/3}$. The linearization of $y(x)$ at 1 is

$$L(x) = y(1) + y'(1)(x-1) = 1 + \frac{1}{3}(x-1) = \frac{1}{3}x + \frac{2}{3}.$$



To illustrate the use of linearizations in making approximations, suppose we want to estimate $\sqrt[3]{1729.03}$. The number 1729.03 is close to $1728 = 12^3$, a perfect cube, so write

$$\sqrt[3]{1729.03} = \sqrt[3]{1728\frac{1729.03}{1728}} = \sqrt[3]{1728}\sqrt[3]{\frac{1729.03}{1728}} = 12\sqrt[3]{\frac{1729.03}{1728}} = 12\sqrt[3]{1 + \frac{1.03}{1728}}.$$

We want to estimate $\sqrt[3]{1+x}$ when $x = 1.03/1728$, a rather small number. The linearization of the function $y(x) = \sqrt[3]{1+x}$ at $x = 0$ is

$$y(0) + y'(0)(x-0) = 1 + \frac{1}{3}(x-0) = 1 + \frac{1}{3}x.$$

Therefore

$$12\sqrt[3]{1+\frac{1.03}{1728}} \approx 12\left(1+\frac{1}{3}\cdot\frac{1.03}{1728}\right) = 12+4\frac{1.03}{1728} = 12.002384\ldots$$

and by comparison the actual value of $\sqrt[3]{1729.03}$ is $12.002383\ldots$, so the linearization of $\sqrt[3]{1+x}$ at 0 gave us an estimate of the cube root of 1729.03 that is correct to 5 digits after the decimal point.

This cube root calculation is based on a story the physicist Richard Feynman told about being challenged to calculate with pencil and paper against an abacus salesman to see who worked faster. See http://www.ee.ryerson.ca/~elf/abacus/feynman.html. Feynman wrote in the middle of his story "I had learned in calculus that for small fractions, the cube root's excess is one-third of the number's excess," which is saying in words that for small x, $\sqrt[3]{1+x} \approx 1+\frac{1}{3}x$ or equivalently $\sqrt[3]{1+x}-1 \approx \frac{1}{3}(1+x-1)$.

Besides their role in making approximations, linearizations are useful in error estimates: they help us estimate the error in $y(x)$ when x undergoes a *small change*. For historical reasons a small change in x is written in calculus as dx instead of Δx, and it is just any small number. The corresponding change in the linearization of y at x is called the *differential* of y and is denoted dy. If x changes by dx and the corresponding change in the linearization of y at x is written as dy then

$$\begin{aligned} dy &= L(x+dx)-L(x) \\ &= (y(x)+y'(x)(x+dx-x))-(y(x)+y'(x)(x-x)) \\ &= (y(x)+y'(x)dx)-y(x) \\ &= y'(x)dx. \end{aligned}$$

Writing $dy = y'(x)dx$ in Leibniz notation makes it $dy = \frac{dy}{dx}dx$, which looks like a rule of fractions. But watch out: in this equation, dy on the left and dy in dy/dx are not the same thing, and likewise the dx in dy/dx and the second factor dx are not the same thing. The whole expression dy/dx is a symbol for the derivative $y'(x)$, while the separate symbol dx is a small change in x and the separate symbol dy is the change in the linearization of y at x *corresponding* to the change by dx in x. The equation $dy = \frac{dy}{dx}dx$ is suggestive, but writing it as $dy = y'(x)dx$ may help in working with this formula.

Let's go back to the previous four examples and write down dy. We just multiply the derivative $y'(x)$ by dx each time.

Example 1. If $y = x^3 - x^2 + x$ then $dy = (3x^2 - 2x + 1)dx$.

Example 2. If $y = \frac{1+2x}{3+4x}$ then $dy = \frac{2}{(3+4x)^2}dx$.

Example 3. If $y = \frac{1}{x}$ then $dy = -\frac{1}{x^2}dx$.

Example 4. If $y = \sqrt[3]{x}$ then $dy = \frac{1}{3}x^{-2/3}dx$.

What do these equations with differentials really mean? They tell us a good estimate for the change in y when x changes by a small amount dx. We will look at the four examples using $dx = .01$ each time.

Example 1. If $y = x^3 - x^2 + x$, then when $x = 1$ and $dx = .01$ we have $dy = (3x^2 - 2x + 1)dx = (3 - 2 + 1)(.01) = .02$. For comparison, the exact change in y when x

changes from 1 to $x + dx = 1.01$ is

$$\Delta y = y(1.01) - y(1) = 1.020201 - 1 = .020201,$$

and $dy = .02$ is a good approximation to this.

If instead $x = 2$ and $dx = .01$ then $dy = (3x^2 - 2x + 1)dx = (3 \cdot 2^2 - 2 \cdot 2 + 1)(.01) = .09$ while the exact change in y when we pass from $x = 2$ to $x + dx = 2.01$ is not far from this:

$$\Delta y = y(2.01) - y(2) = .0905.$$

Example 2. If $y = \frac{1+2x}{3+4x}$, then when $x = 0$ and $dx = .01$ we have $dy = \frac{2}{(3+4x)^2}dx = \frac{2}{3^2}(.01) = .0022\ldots$. The exact change in y when we move from $x = 0$ to $x + dx = .01$ is

$$\Delta y = y(.01) - y(0) = \frac{1.02}{3.04} - \frac{1}{3} = .00219\ldots,$$

and $dy = .0022\ldots$ is a good approximation to Δy.

Example 3. If $y = \frac{1}{x}$, then when $x = 2$ and $dx = .01$ we have $dy = -\frac{1}{x^2}dx = -\frac{1}{4}(.01) = -.0025$. The exact change in y when x changes from 2 to $x + dx = 2.01$ is

$$\Delta y = y(2.01) - y(2) = \frac{1}{2.01} - \frac{1}{2} = -.00248\ldots,$$

and the differential $dy = -.0025$ approximates this well.

Example 4. If $y = \sqrt[3]{x}$, then when $x = 1$ and $dx = .01$ we have $dy = \frac{1}{3}x^{-2/3}dx = \frac{1}{3}(1)(.01) = .00333\ldots$. The exact change in y when x changes from 1 to $x + dx = 1.01$ is

$$\Delta y = y(1.01) - y(1) = \sqrt[3]{1.01} - \sqrt[3]{1} = .003322\ldots,$$

which is approximated well by $dy = .00333\ldots$