DERIVATIVE RULES FROM APPROXIMATIONS

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When a function $f(x)$ has a derivative $f'(x)$, at each x we can say for very small h that

$$\frac{f(x+h)-f(x)}{h} \approx f'(x),$$

and if we multiply both sides by x and add h to both sides we get the *fundamental linear approximation*

$$f(x+h) \approx f(x) + f'(x)h \text{ for small } h.$$

This is called a linear approximation because the right side is a linear polynomial in h. Here $f'(x)$ appears as the linear coefficient in an approximation. Turning this around, if there is a number k such that $f(x+h) \approx f(x) + kh$ for all small h then the number k has to be $f'(x)$. We will use this idea to explain basic rules of differentiation.

Sum rule: Let $a(x) = f(x) + g(x)$. (Use a for addition.) For small h we have

$$\begin{aligned}
a(x+h) &= f(x+h) + g(x+h) \\
&\approx (f(x) + f'(x)h) + (g(x) + g'(x)h) \\
&= (f(x) + g(x)) + (f'(x) + g'(x))h \\
&= a(x) + (f'(x) + g'(x))h.
\end{aligned}$$

Since the coefficient of h in this approximation is $f'(x) + g'(x)$, we see that $(f(x) + g(x))' = a'(x) = f'(x) + g'(x)$.

Difference rule: Let $s(x) = f(x) - g(x)$. (Use s for subtraction.) For small h,

$$\begin{aligned}
s(x+h) &= f(x+h) - g(x+h) \\
&\approx (f(x) + f'(x)h) - (g(x) + g'(x)h) \\
&= (f(x) - g(x)) + (f'(x) - g'(x))h \\
&= s(x) + (f'(x) - g'(x))h.
\end{aligned}$$

The coefficient of h in this approximation is $f'(x) - g'(x)$, so $(f(x) - g(x))' = s'(x) = f'(x) - g'(x)$.

Constant multiple rule: Let $m(x) = cf(x)$ for a constant c. For small h,

$$\begin{aligned}
m(x+h) &= cf(x+h) \\
&\approx c(f(x) + f'(x)h) \\
&= cf(x) + cf'(x)h \\
&= m(x) + cf'(x)h.
\end{aligned}$$

The coefficient of h is $cf'(x)$, so $(cf(x))' = m'(x) = cf'(x)$.

Product rule: Let $p(x) = f(x)g(x)$. (Use p for product.) For small h,

$$\begin{aligned} p(x+h) &= f(x+h)g(x+h) \\ &\approx (f(x)+f'(x)h)(g(x)+g'(x)h) \\ &= f(x)g(x)+f(x)g'(x)h+f'(x)hg(x)+g(x)g'(x)h^2 \\ &= f(x)g(x)+(f(x)g'(x)+f'(x)g(x))h+g(x)g'(x)h^2 \\ &= p(x)+(f(x)g'(x)+f'(x)g(x))h+g(x)g'(x)h^2. \end{aligned}$$

Here there is an h-term and an h^2-term. For small h the h^2-term is negligible by comparison to the h-term. Since the coefficient of h is $f(x)g'(x) + f'(x)g(x)$, we have $(f(x)g(x))' = p'(x) = f(x)g'(x) + f'(x)g(x)$. Therefore the form of the product rule can be regarded as a consequence of the distributive law for multiplication over addition (or FOIL, as it is often called by students in the US).

Quotient rule: We will not use approximations to explain the quotient rule for $(f(x)/g(x))'$, but rather use the product rule by viewing $f(x)/g(x)$ as a product:

$$\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}.$$

First we will compute the derivative of $1/g(x)$ when $g(x) \neq 0$, using the limit definition of the derivative:

$$\begin{aligned} \left(\frac{1}{g(x)}\right)' &= \lim_{h\to 0} \frac{1/g(x+h) - 1/g(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{g(x+h)} - \frac{1}{g(x)}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\frac{g(x)-g(x+h)}{g(x+h)g(x)} \\ &= \lim_{h\to 0} \frac{g(x)-g(x+h)}{h}\frac{1}{g(x+h)g(x)} \\ &= \lim_{h\to 0} \frac{-(g(x+h)-g(x))}{h}\frac{1}{g(x+h)g(x)}. \end{aligned}$$

In the denominator, $g(x+h) \to g(x)$ as $h \to 0$ since differentiability at x implies continuity at x. Therefore we have found the derivative of $1/g(x)$:

$$\left(\frac{1}{g(x)}\right)' = \lim_{h\to 0} \frac{-(g(x+h)-g(x))}{h}\frac{1}{g(x)^2} = -g'(x)\frac{1}{g(x)^2} = \frac{-g'(x)}{g(x)^2}.$$

Now we can compute $(f(x)/g(x))'$ with the product rule:

$$\begin{aligned} \left(\frac{f(x)}{g(x)}\right)' &= \left(f(x)\cdot\frac{1}{g(x)}\right)' \\ &= f(x)\left(\frac{1}{g(x)}\right)' + \frac{1}{g(x)}f'(x) \\ &= f(x)\left(\frac{-g'(x)}{g(x)^2}\right) + \frac{f'(x)}{g(x)} \\ &= \frac{-f(x)g'(x)}{g(x)^2} + \frac{f'(x)g(x)}{g(x)^2} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}. \end{aligned}$$