GRAPHS OF FUNCTIONS AND DERIVATIVES

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We will review here some of the terminology and results associated with graphs where first and second derivatives are helpful.

1. The shape of a graph

Definition 1.1. A value of a function, $f(c)$, is called

(1) a **local maximum value** if it's larger than values of $f(x)$ at all x close to c,
(2) a **local minimum value** if it's smaller than values of $f(x)$ at all x close to c,
(3) an **absolute maximum value** if it's the greatest value of $f(x)$,
(4) an **absolute minimum value** if it's the least value of $f(x)$.

For example, in Figure 1 below is a graph defined on all numbers. The function has local maximum values at a and b, and a local minimum value at 0. Its absolute maximum value is at a and its absolute minimum value is at 0.

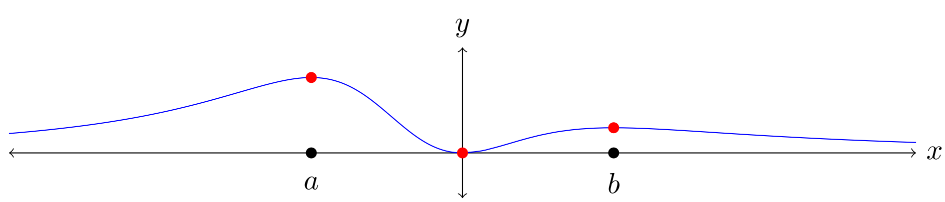


Figure 1. A graph with local maxima and minima marked.

In Figure 2, on a closed interval $[a, b]$ the absolute maximum and minimum values of the function graphed there are at the endpoints. There is a local maximum value at a and a local minimum value at b.

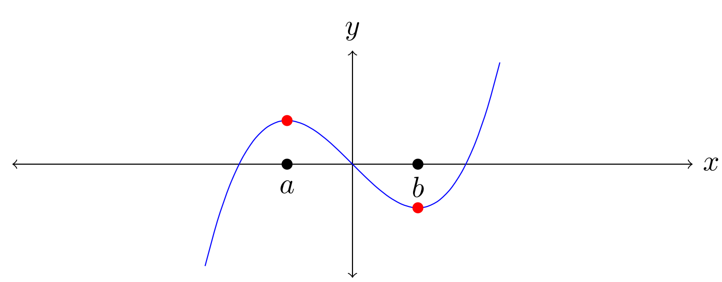


Figure 2. A graph with local maxima and minima marked.

In Figure 3, the graph has a vertical asymptote at $x = a$ and no absolute maximum or minimum values: near any number besides a, the function has a larger value and a smaller value.

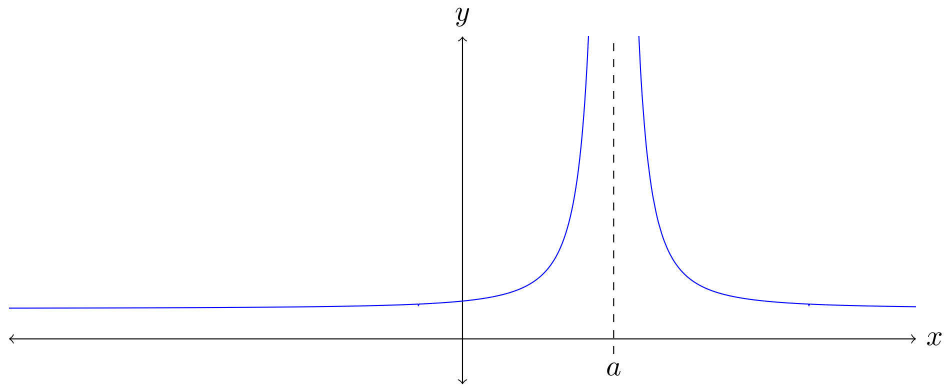


FIGURE 3. A graph with no local maximum or minimum values.

The following theorem provides conditions under which a function is guaranteed to have an absolute maximum value and an absolute minimum value somewhere (called its "extreme values").

Theorem 1.2 (Extreme Value Theorem). *A continuous function on a closed and bounded interval* $[a, b]$ *has absolute maximum and minimum values on this interval.*

If any assumptions are removed, the theorem has counterexamples. Figure 4 is a *discontinuous* function on a closed and bounded interval $[a, b]$ with no absolute maximum or minimum values on $[a, b]$, Figure 5 is a continuous function on an *open* bounded interval (a, b) with a vertical asymptote at both endpoints, and Figure 6 is a continuous function on a closed and *unbounded* interval $[a, \infty)$. In all cases there is no absolute maximum value and no absolute minimum value.

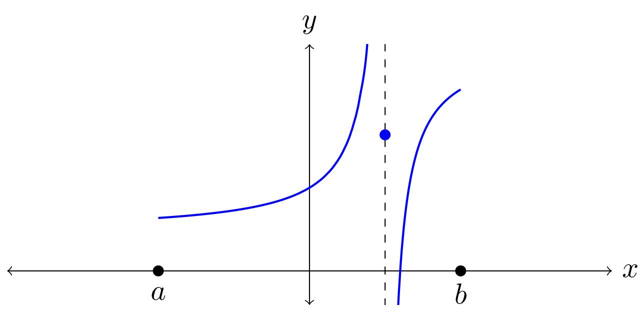


FIGURE 4. A discontinuous function on $[a, b]$.

Definition 1.3. We say that a function $f(x)$ defined on an interval is

(1) **increasing** if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ on the whole interval,
(2) **decreasing** if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ on the whole interval,
(3) **concave up** if the graph is above all its tangent lines near the points of tangency,
(4) **concave down** if the graph is below all its tangent lines near the points of tangency.

In Figure 7, the graph is concave up for $x < 0$ (see green tangent line) and concave down for $x > 0$ (see red tangent line).

Figure 8 is a typical illustration of everywhere concave up and concave down curves: the parabola on the left is concave up everywhere while the parabola on the right is concave

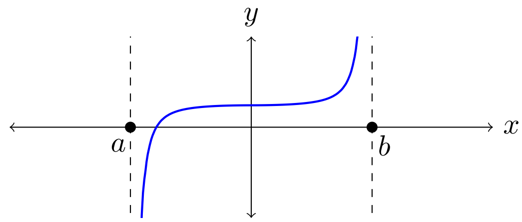


FIGURE 5. A continuous function on (a, b).

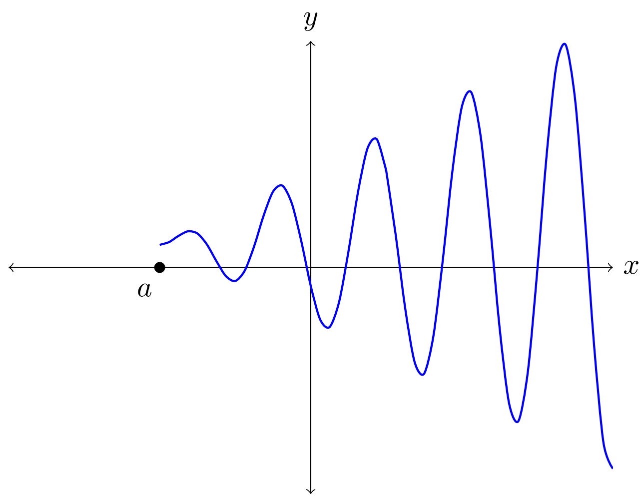


FIGURE 6. A continuous function on $[a, \infty)$.

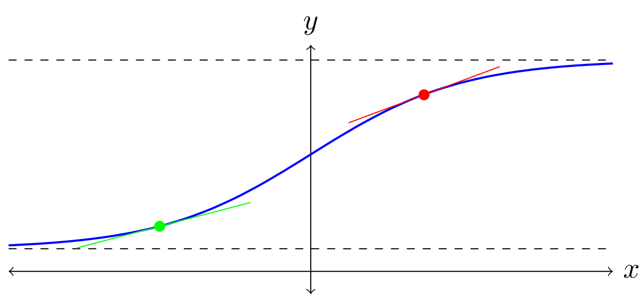


FIGURE 7. A graph that is concave up and concave down.

down everywhere. The similarity of the shape of the parabola on the left to $\cup$, which looks like the first letter in "up", can help you remember that this curve is concave *up*.

Definition 1.4. A point on a graph is an **inflection point** if the graph is concave up on one side of it and concave down on the other side of it.

Inflection points on the graph in Figure 9 are marked in red. The graph is concave down between the inflection points and concave up elsewhere.

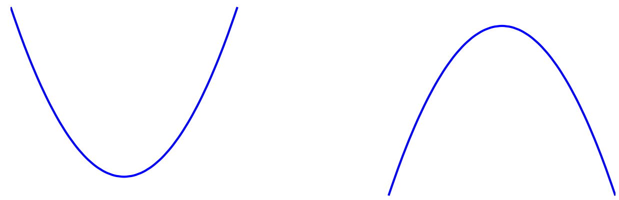


FIGURE 8. Typical concave up (left) and concave down (right) shapes.

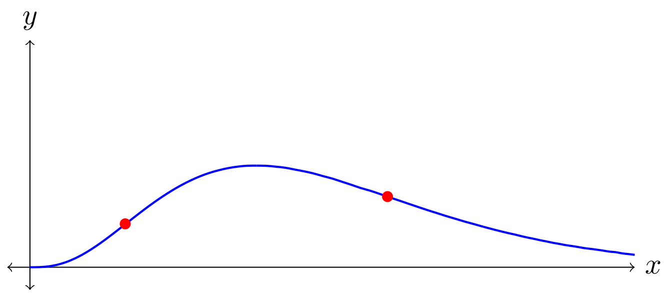


FIGURE 9. Inflection points on a graph.

2. Derivatives

The concepts introduced earlier (local maximum and minimum values, increasing, decreasing, concavity, inflection points) are geometric. We now connect them with the behavior of derivatives.

Theorem 2.1 (Fermat's Theorem). *If $f(c)$ is a local maximum or minimum value of a differentiable function $f(x)$ on an open interval then $f'(c) = 0$.*

This tells us **where** to look to find absolute or local maximum/minimum values of $f(x)$ on an open interval: (i) where $f'(x) = 0$ and (ii) where $f'(x)$ does not exist. When looking for absolute maximum/minimum values on an interval that includes endpoints, we should check $f(x)$ at the endpoints too. All these cases can be relevant. In Figure 10, the absolute maximum values are at the endpoints and the absolute minimum value is at a solution of $f'(x) = 0$. In Figure 11, the absolute maximum values are at the endpoints and the absolute minimum value is where $f'(x)$ does not exist.

Example 2.2. We will use calculus to determine the absolute maximum and minimum values of $f(x) = x^3 - x$ on the interval [0,1.5]. It is graphed in Figure 12.

The function $x^3 - x$ is differentiable on the interval $(0, 1.5)$, with derivative $3x^2 - 1$, so the absolute maximum and minimum values of $x^3 - x$ on $[0, 1.5]$ can be found either where $3x^2 - 1 = 0$ or at its endpoints. The solutions of $3x^2 - 1 = 0$ are $x = \pm 1/\sqrt{3}$, and in $[0, 1.5]$ the only relevant choice is $x = 1/\sqrt{3} \approx .577$.

- At $x = 1/\sqrt{3}$, $f(1/\sqrt{3}) = -2/(3\sqrt{3}) \approx -.384$.
- At the endpoints, $f(0) = 0$ and $f(1.5) = 1.875$.

Comparing these values, the absolute maximum value of $x^3 - x$ on $[0, 1.5]$ is 1.875 and the absolute minimum value is $-2/(3\sqrt{3}) \approx -.384$.

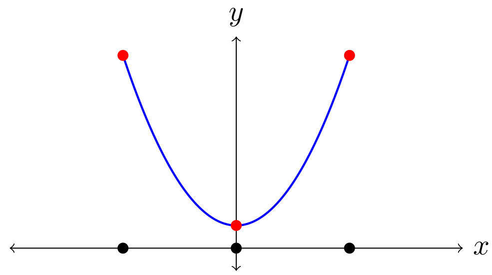


FIGURE 10. Absolute maximum and minimum values at endpoints and where $f'(x) = 0$.

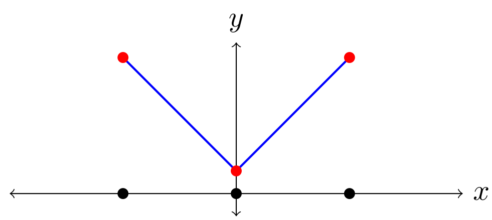


FIGURE 11. Absolute maximum and minimum values at endpoints and where $f'(x)$ does not exist.

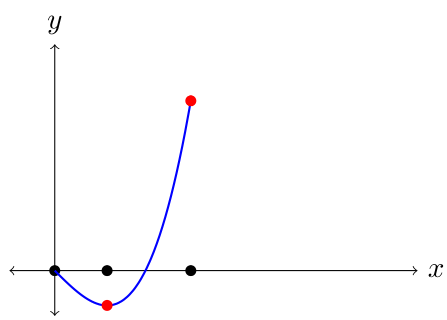


FIGURE 12. The graph of $y = x^3 - x$ on [0,1.5].

Warning: For a differentiable function $f(x)$, any place where it has a local maximum or minimum value on an open interval must satisfy $f'(x) = 0$. However, solutions to $f'(x) = 0$ are **NOT** necessarily where $f(x)$ has a local maximum or minimum value, *e.g.*, $f(x) = x^3$ at $x = 0$. See Figure 13. This graph is increasing everywhere.

Theorem 2.3. *Let $f(x)$ be a differentiable function.*

(1) *Increasing/Decreasing Test:*
- *If $f'(x) > 0$ on an interval then $f(x)$ is increasing on that interval.*
- *If $f'(x) < 0$ on an interval then $f(x)$ is decreasing on that interval.*

(2) *Concave Up/Down Test*
- *If $f''(x) > 0$ on an interval then $f(x)$ is concave up on that interval.*
- *If $f''(x) < 0$ on an interval then $f(x)$ is concave down on that interval.*

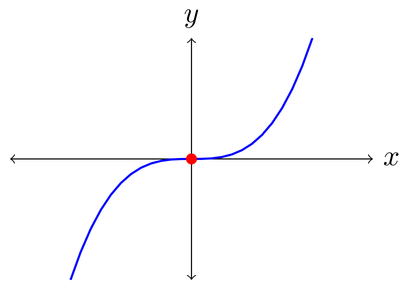


FIGURE 13. The graph of $y = x^3$: $y'(0) = 0$ and no local maximum or minimum at $x = 0$.

Example 2.4. In Figure 14 is a graph of $y = f(x) = x^3 - 3x^2 + x$. We will use calculus to determine the open intervals where $f(x)$ is increasing and decreasing, and where its graph is concave up and concave down.

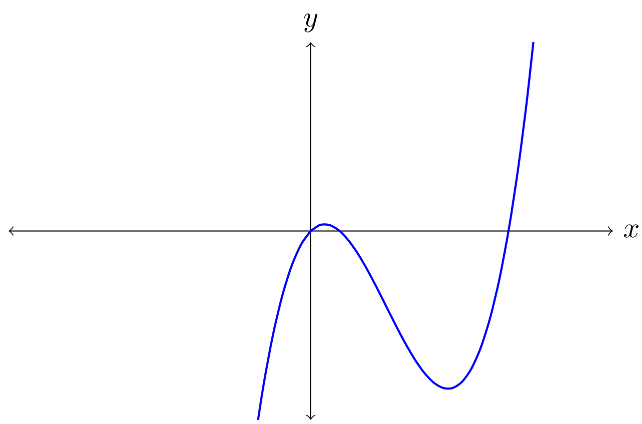


FIGURE 14. The graph of $y = x^3 - 3x^2 + x$.

We will break up our task into four steps: (1) determine where $f'(x) = 0$, (2) determine intervals where $f'(x) > 0$ and $f'(x) < 0$, (3) determine where $f''(x) = 0$, and (4) determine intervals where $f''(x) > 0$ and $f''(x) < 0$.

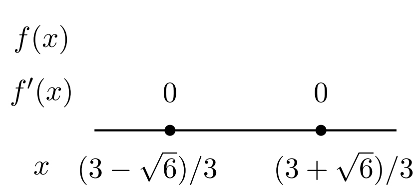
Step 1: Determine where $f'(x) = 0$.

The first derivative of $f(x) = x^3 - 3x^2 + x$ is $f'(x) = 3x^2 - 6x + 1$, and

$$f'(x) = 0 \Longleftrightarrow 3x^2 - 6x + 1 = 0 \Longleftrightarrow x = \frac{6 \pm \sqrt{6^2 - 4 \cdot 3 \cdot 1}}{2 \cdot 3} = \frac{6 \pm \sqrt{24}}{6} = \frac{3 \pm \sqrt{6}}{3},$$

so $f'(x) = 0 \Longleftrightarrow x = (3 + \sqrt{6})/3 \approx 1.816$ and $x = (3 - \sqrt{6})/3 \approx .1835$.

Make a first derivative chart, shown below, with a row for x-values under the number line and rows for f' and f above the number line. Mark the critical numbers $(3 \pm \sqrt{6})/3$ below the number line and mark f' as 0 at these numbers.



Step 2: Determine intervals where $f'(x) > 0$ and $f'(x) < 0$.

The critical numbers divide the number line into 3 open intervals: $(-\infty, (3-\sqrt{6})/3)$, $((3-\sqrt{6})/3, (3+\sqrt{6})/3)$, and $((3+\sqrt{6})/3, \infty)$. On each of these intervals the first derivative has a constant sign, and we can determine the sign by computing the derivative at any number in the interval.

- In $(-\infty, (3-\sqrt{6})/3)$ use $x = 0$: $f'(0) = 1 > 0$, so $f'(x) > 0$ on this interval.
- In $((3-\sqrt{6})/3, (3+\sqrt{6})/3)$ use $x = 1$: $f'(1) = -2 < 0$, so $f'(x) < 0$ on this interval.
- In $((3+\sqrt{6})/3, \infty)$ use $x = 2$: $f'(2) = 1 > 0$, so $f'(x) > 0$ on this interval.

Now we go back to the first derivative chart and fill in the signs of $f'(x)$ for each interval as positive $(+)$ or negative $(-)$, and then draw an arrow above the sign to indicate if $f(x)$ is increasing $(\nearrow)$ or decreasing $(\searrow)$ based on the sign of the first derivative. See below.

$f(x)$	$\nearrow$		$\searrow$		$\nearrow$
$f'(x)$	$+$	0	$-$	0	$+$
x		$(3-\sqrt{6})/3$		$(3+\sqrt{6})/3$	

From this chart, $x^3 - 3x^2 + x$ is increasing on $(-\infty, (3-\sqrt{6})/3)$ and $((3+\sqrt{6})/3, \infty)$ and decreasing on $((3-\sqrt{6})/3, (3+\sqrt{6})/3)$.

Step 3: Determine where $f''(x) = 0$.

We find where $f''(x) = 0$:

$$f'(x) = 3x^2 - 6x + 1 \Longrightarrow f''(x) = 6x - 6,$$

so $f''(x) = 0 \Longleftrightarrow 6x - 6 = 0 \Longleftrightarrow 6x = 6 \Longleftrightarrow x = 1$. We start a second derivative chart below, with x-values under the number line and rows for f'' and f above the number line. Mark $x = 1$ and place a 0 on the row for f'' above $x = 1$.

$f(x)$			
$f''(x)$		0	
x		1	

Step 4: Determine intervals where $f''(x) > 0$ and $f''(x) < 0$.

The second derivative has constant sign on $(-\infty, 1)$ and $(1, \infty)$, and the choice of sign can be found by computing $f''(x)$ at any number in each interval.

- In $(-\infty, 1)$ use $x = 0$: $f''(0) = -6 < 0$, so $f''(x) < 0$ on this interval.
- In $(1, \infty)$ use $x = 2$: $f''(2) = 6 > 0$, so $f''(x) > 0$ on this interval.

Now we can go back and fill in the second derivative chart, indicating in the row for $f''(x)$ where $f''(x)$ is positive $(+)$ and negative $(-)$, after which we can indicate in the row for $f(x)$ where the graph is concave up $(\cup)$ and concave down $(\cap)$. See below.

$f(x)$	$\cap$		$\cup$
$f''(x)$	$-$	0	$+$
x		1	

Thus the graph of $y = x^3 - 3x^2 + x$ is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

Example 2.5. Let $f(x) = x/(x^3 - 1)$. A graph of $y = f(x)$ is in Figure 15. We want to find where the function is increasing and decreasing, and where its graph is concave up and concave down.

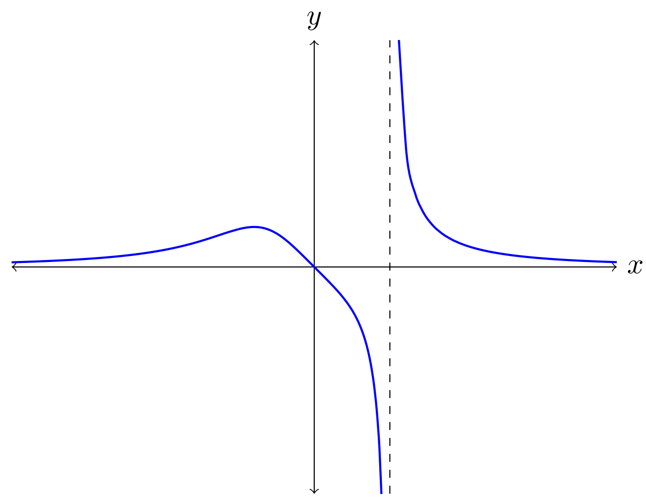


FIGURE 15. The graph of $y = x/(x^3 - 1)$.

We can proceed as before, in four steps: (1) determine where $f'(x) = 0$, (2) determine intervals where $f'(x) > 0$ and $f'(x) < 0$, (3) determine where $f''(x) = 0$, and (4) determine intervals where $f''(x) > 0$ and $f''(x) < 0$.

A difference from the previous example is that $f(x)$ has a vertical asymptote at $x = 1$, so we **need to include** $x = 1$ as a point in both derivative charts. We will streamline the four steps by combining the first two steps and last two steps.

Steps 1 and 2: Determine where $f'(x) = 0$ and intervals where $f'(x) > 0$ and $f'(x) < 0$. By the quotient rule,

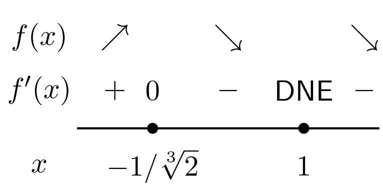
$$f'(x) = \frac{(x^3 - 1) - x(3x^2)}{(x^3 - 1)^2} = \frac{-1 - 2x^3}{(x^3 - 1)^2},$$

so $f'(x) = 0 \iff -1 - 2x^3 = 0 \iff 2x^3 = -1 \iff x = -1/\sqrt[3]{2} \approx -.79$. The critical numbers are $x = -1/\sqrt[3]{2}$ (where $f'(x) = 0$) and $x = 1$ (where $f'(x)$ doesn't exist).

On the intervals $(-\infty, -1/\sqrt[3]{2})$, $(-1/\sqrt[3]{2}, 1)$, and $(1, \infty)$ the derivative has constant sign, which we can determine by sampling $f'(x)$ at one point in each interval.

- In $(-\infty, -1/\sqrt[3]{2})$ use $x = -1$: $f'(-1) = 1/4 > 0$, so $f'(x) > 0$ on this interval.
- In $(-1/\sqrt[3]{2}, 1)$ use $x = 0$: $f'(0) = -1 < 0$, so $f'(x) < 0$ on this interval.
- In $(1, \infty)$ use $x = 2$: $f'(2) = -17/49 < 0$, so $f'(x) < 0$ on this interval.

We collect this information into the first derivative chart below.



Thus $f(x)$ is increasing on $(-\infty, -1/\sqrt[3]{2})$ and decreasing on $(-1/\sqrt[3]{2}, 1)$ and on $(1, \infty)$.

Steps 3 and 4: Determine where $f''(x) = 0$ and intervals where $f''(x) > 0$ and $f''(x) < 0$.

Differentiating the formula for $f'(x)$,

$$\begin{aligned} f''(x) &= \left(\frac{-1-2x^3}{(x^3-1)^2}\right)' \\ &= \frac{(x^3-1)^2(-6x^2)-(-1-2x^3)2(x^3-1)(3x^2)}{(x^3-1)^4} \\ &= \frac{-6x^2(x^3-1)+6x^2(1+2x^3)}{(x^3-1)^3} \\ &= \frac{6x^5+12x^2}{(x^3-1)^3} \\ &= \frac{6x^2(x^3+2)}{(x^3-1)^3}. \end{aligned}$$

so $f''(x)=0 \Longleftrightarrow x^2(x^3+2)=0 \Longleftrightarrow x=0$ or $x^3=-2 \Longleftrightarrow x=0$ or $x=-\sqrt[3]{2}\approx -1.26$.

The intervals where $f''(x)$ has constant sign are separated by $x=-\sqrt[3]{2}$ and $x=0$ (where $f''(x)=0$) and $x=1$ (where $f''(x)$ doesn't exist).

- In $(-\infty,-\sqrt[3]{2})$ use $x=-2$: $f''(-2)=16/81>0$, so $f''(x)>0$ on this interval.
- In $(-\sqrt[3]{2},0)$ use $x=-1$: $f''(-1)=-3/4<0$, so $f''(x)<0$ on this interval.
- In $(0,1)$ use $x=1/2$: $f''(1/2)=-1632/343<0$, so $f''(x)<0$ on this interval.
- In $(1,\infty)$ use $x=2$: $f''(2)=240/343>0$, so $f''(x)>0$ on this interval.

Here is a second derivative table based on this information.

$f(x)$	∪		∩		∩		∪
$f''(x)$	+	0	−	0	−	DNE	+
x		$-\sqrt[3]{2}$		0		1	

Therefore $f(x)$ is concave up on $(-\infty,-\sqrt[3]{2})$ and $(1,\infty)$ and it is concave down on $(-\sqrt[3]{2},0)$ and $(0,1)$.

Note: Although $f''(x)=0$ at $x=-\sqrt[3]{2}$ and at $x=0$, there is only an inflection point where $x=-\sqrt[3]{2}$: the sign of $f''(x)$ is negative on both sides of $x=0$, so $(0,0)$ is not an inflection point.

Example 2.6. For $n=2,3,4,\ldots$ set $f(x)=x^ne^{-x}$ for $x>0$. See Figure 16. Let's find the open intervals where $f(x)$ is increasing and decreasing, and where its graph is concave up and concave down. The function $f(x)$ depends on n, so we expect answers to depend on n.

The first derivative of $f(x)$ is

$$f'(x)=(x^n)'e^{-x}+x^n(e^{-x})'=nx^{n-1}e^{-x}-x^ne^{-x}=(nx^{n-1}-x^n)e^{-x}=x^{n-1}(n-x)e^{-x},$$

so $f'(x)=0$ if and only if $x=n$ (we only use $x>0$). This leads to the chart below, where for $0<x<n$ we have $f'(x)>0$ since $f'(1)=e^{-1}(n-1)>0$ and for $x>n$ we have $f'(x)<0$ since $f'(n+1)=(n+1)^{n-1}e^{-(n+1)}(n-(n+1))=(n+1)^{n-1}e^{-(n+1)}(-1)<0$.

$f(x)$	↗		↘
$f'(x)$	+	0	−
x		n	

Thus $f(x)$ has a global maximum value at $x=n$.

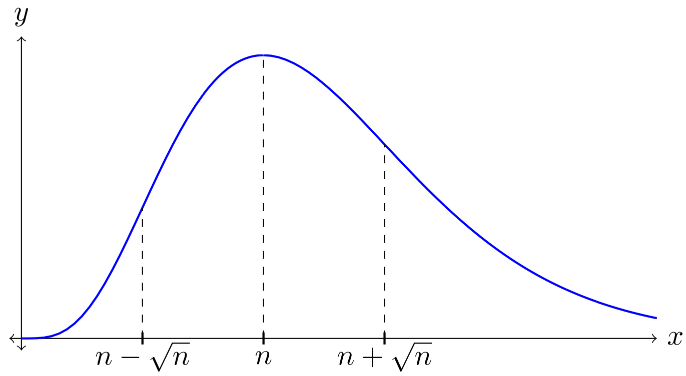


FIGURE 16. The graph of $y = x^n e^{-x}$ for $n \geq 2$.

The second derivative of $f(x)$ is

$$\begin{aligned} f''(x) &= ((nx^{n-1} - x^n)e^{-x})' \\ &= (nx^{n-1} - x^n)'e^{-x} + (nx^{n-1} - x^n)(e^{-x})' \\ &= (n(n-1)x^{n-2} - nx^{n-1})e^{-x} - (nx^{n-1} - x^n)e^{-x} \\ &= (n(n-1)x^{n-2} - 2nx^{n-1} + x^n)e^{-x} \\ &= = x^{n-2}(x^2 - 2nx + n(n-1))e^{-x}, \end{aligned}$$

so we have $f''(x) = 0 \Longleftrightarrow x^2 - 2nx + n(n-1) = 0$, which by the quadratic formula is the same as

$$x = \frac{2n \pm \sqrt{(2n)^2 - 4n(n-1)}}{2} = n \pm \sqrt{n^2 - n(n-1)} = n \pm \sqrt{n},$$

which are both positive. A sample number greater than $n + \sqrt{n}$ is $2n$ and

$$f''(2n) = (2n)^{n-2}((2n)^2 - 2n(2n) + n(n-1))e^{-2n} = (2n)^{n-2}(n(n-1))e^{-2n} > 0.$$

A sample number between $n - \sqrt{n}$ and $n + \sqrt{n}$ is n and

$$f''(n) = n^{n-2}(n^2 - 2n^2 + n(n-1))e^{-n} = n^{n-2}(-n)e^{-n} < 0.$$

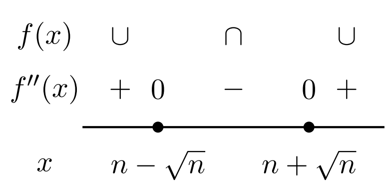
For $n \geq 3$, a sample number between 0 and $n - \sqrt{n}$ is 1 and

$$f''(1) = (1 - 2n + n(n-1))e^{-1} = (n^2 - 3n + 1)e^{-1} > 0.$$

For $n = 2$, a sample number between 0 and $n - \sqrt{n} = 2 - \sqrt{2} \approx .58$ is $1/2$ and

$$f''(x) = x^0(x^2 - 4x + 2)e^{-x} \Longrightarrow f''(1/2) = (1/4 - 2 + 2)e^{-1/2} = (1/4)e^{-1/2} > 0.$$

This explains the second derivative chart below for $f(x)$ when $x > 0$, which shows $f(x)$ has inflection points at $n \pm \sqrt{n}$.



Example 2.7. For $a > 0$, let's find the open intervals where $f(x) = e^{-ax^2}$ is increasing and decreasing, and where its graph is concave up and concave down. See Figure 17. Since the function depends on a parameter a, we should expect that our answers may depend on a.

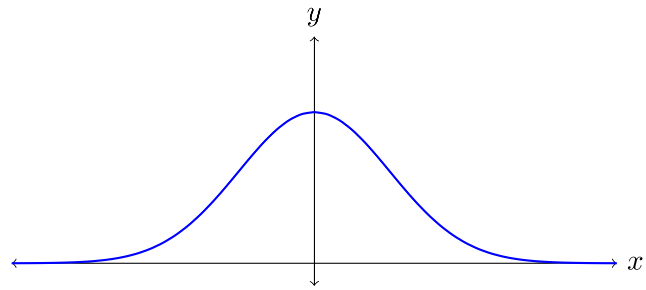


FIGURE 17. The graph of $y = e^{-ax^2}$.

The first and second derivatives are

$$f'(x) = -2axe^{-ax^2}, \quad f''(x) = -2ax(-2axe^{-ax^2}) - 2ae^{-ax^2} = (4a^2x^2 - 2a)e^{-ax^2}.$$

For the first derivative,

$$f'(x) = 0 \Longleftrightarrow x = 0,$$

and this leads to the first derivative chart below: if $x < 0$ then $f'(x) > 0$ (try $x = -1$), and if $x > 0$ then $f'(x) < 0$ (try $x = 1$).

$f(x)$	$\nearrow$		$\searrow$
$f'(x)$	$+$	0	$-$
x		0	

Thus e^{-ax^2} is increasing on the interval $(-\infty, 0)$ and decreasing on the interval $(0, \infty)$.

For the second derivative,

$$f''(x) = 0 \Longleftrightarrow 4a^2x^2 - 2a = 0 \Longleftrightarrow x^2 = \frac{2a}{4a^2} = \frac{1}{2a} \Longleftrightarrow x = \pm\frac{1}{\sqrt{2a}}.$$

The intervals where $f''(x)$ has constant sign are $(-\infty, -1/\sqrt{2a})$, $(-1/\sqrt{2a}, 1/\sqrt{2a})$, and $(1/\sqrt{2a}, \infty)$. Since $f''(x) = (4a^2x^2 - 2a)e^{-ax^2}$ and e^{-ax^2} is always positive, the sign of $f''(x)$ on an interval is the same as the sign of $4a^2x^2 - 2a = 2a(2ax^2 - 1)$ on that interval.

- If x is in $(-\infty, -1/\sqrt{2a})$ or $(1/\sqrt{2a}, \infty)$ then $x^2 > 1/2a$, so $2ax^2 - 1 > 0$, so $f''(x) > 0$.
- If x is in $(-1/\sqrt{2a}, 1/\sqrt{2a})$ then $x^2 < 1/2a$, so $2ax^2 - 1 < 0$ (or just test $f''(0) = -2a < 0$), so $f''(x) < 0$.

This information leads to the following second derivative chart.

$f(x)$	$\cup$		$\cap$		$\cup$
$f''(x)$	$+$	0	$-$	0	$+$
x		$-1/\sqrt{2a}$		$1/\sqrt{2a}$	

From this chart, e^{-ax^2} has inflection points at $x = \pm 1/\sqrt{2a}$ and it is concave up for x in $(-\infty, -1/\sqrt{2a})$ or $(1/\sqrt{2a}, \infty)$ and it is concave down for x in $(-1/\sqrt{2a}, 1\sqrt{2a})$.

3. Caution!

Any place where the graph of $y = f(x)$ has an inflection point must satisfy $f''(x) = 0$, but the graph does **NOT** have to have an inflection point when $f''(x) = 0$, *e.g.*, $f(x) = x^4$ at $x = 0$. See Figure 18. This graph is concave up everywhere.

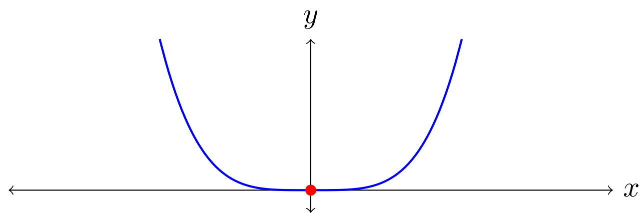


Figure 18. The graph of $y = x^4$: $y''(0) = 0$ and no inflection point at $(0, 0)$.