CHAIN RULE PROBLEMS

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The chain rule says $(f(g(x)))' = f'(g(x))g'(x)$, or $(f(u))' = f'(u)u'(x)$ if $u = g(x)$. To carry out the chain rule, know basic derivatives well so you can build on that. In the table below, observe how each basic derivative in the first column generalizes by the chain rule with $g(x)$ in place of x in the third column.

Basic derivative	**Composition**	**Chain rule example**
$(x^c)' = cx^{c-1}$ for constant c	$g(x)^c = u^c$ for $u = g(x)$ and constant c	$(g(x)^c)' = cg(x)^{c-1}g'(x)$ for constant c
$(x^3)' = 3x^2$	$g(x)^3 = u^3$ for $u = g(x)$	$((g(x))^3)' = 3g(x)^2g'(x)$
$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$	$\sqrt{g(x)} = \sqrt{u}$ for $u = g(x)$	$(\sqrt{g(x)})' = \frac{1}{2\sqrt{g(x)}}g'(x)$
$\left(\frac{1}{x}\right)' = \frac{-1}{x^2}$	$\frac{1}{g(x)} = \frac{1}{u}$ for $u = g(x)$	$\left(\frac{1}{g(x)}\right)' = \frac{-1}{g(x)^2}g'(x)$
$(e^x)' = e^x$	$e^{g(x)} = e^u$ for $u = g(x)$	$(e^{g(x)})' = e^{g(x)}g'(x)$
$(\ln x)' = \frac{1}{x}$	$\ln g(x) = \ln u$ for $u = g(x)$	$(\ln g(x))' = \frac{1}{g(x)}g'(x)$
$(\sin x)' = \cos x$	$\sin(g(x)) = \sin u$ for $u = g(x)$	$(\sin(g(x)))' = \cos(g(x))g'(x)$
$(\cos x)' = -\sin x$	$\cos(g(x)) = \cos u$ for $u = g(x)$	$(\cos(g(x)))' = -\sin(g(x))g'(x)$
$(\tan x)' = \sec^2 x$	$\tan(g(x)) = \tan u$ for $u = g(x)$	$(\tan(g(x)))' = \sec^2(g(x))g'(x)$
$(\arctan x)' = \frac{1}{x^2+1}$	$\arctan(g(x)) = \arctan u$ for $u = g(x)$	$(\arctan(g(x)))' = \frac{1}{g(x)^2+1}g'(x)$

Problems.

(1) $\sin^2 x$ (This *means* $(\sin x)^2$, so the "outer" function is x^2, *not* $\sin x$.)
(2) $\sin(\sin x)$
(3) $\sin(\sin(\sin x))$
(4) $\sin(\sin(\sin(\sin x)))$
(5) $\sqrt{x^2+5x}$
(6) $(x^3+x^2+1)^4$
(7) $e^{-x^2/5}$
(8) $(\sqrt{x}+x^3)^8$
(9) $\dfrac{1}{\cos x}$
(10) $\cos\left(\dfrac{1}{x}\right)$
(11) $3^{\cos x}$
(12) $\cos(5x)$
(13) $3^{\cos(5x)}$
(14) $\cos(3^x)$
(15) $\cos(\sqrt{x})$
(16) $\sqrt{\cos x}$
(17) $\ln(4x+3)$
(18) $\dfrac{1}{(\ln x)^4}$
(19) $\tan(e^{-x}+4x^2)$.
(20) $\tan((2x+1)^3+x)$
(21) $\tan^3 x$ (This *means* $(\tan x)^3$, so the "outer" function is x^3, *not* $\tan x$.)
(22) $\sin(x^3-x)$
(23) $((x^2-x)^3+x+2)^4$
(24) $\dfrac{1}{x^3+3^x}$.
(25) $\cos(\ln(x^3+1))$
(26) $\ln(\cos(x^3+1))$
(27) $(\ln(\cos x))^3+1$
(28) $e^{\sin(x^2)}$
(29) $\sin(e^{x^2})$
(30) $\sin^2(e^x)$. (This *means* $(\sin(e^x))^2$, so the "outer" function is x^2, *not* $\sin x$.)

Answers.

(1) $2(\sin x)(\cos x)$. (Let $u = \sin x$ to make the function u^2.)

(2) $(\cos(\sin x))\cos x$. (Let $u = \sin x$ to make the function $\sin(u)$.)

(3) $(\cos(\sin(\sin x)))\cos(\sin x)\cos x$. (Let $u = \sin(\sin x)$ to make the function $\sin(u)$.)

(4) $(\cos(\sin(\sin(\sin x))))(\cos(\sin(\sin x)))\cos(\sin x)\cos x$. (Let $u = \sin(\sin(\sin x))$ to make the function $\sin(u)$.)

(5) $\dfrac{1}{2\sqrt{x^2+5x}}(2x+5)$. (Let $u = x^2 + 5x$ to make the function $\sqrt{u} = u^{1/2}$.)

(6) $4(x^3+x^2+1)^3(3x^2+2x)$. (Let $u = x^3 + x^2 + 1$ to make the function u^4.)

(7) $e^{-x^2/5}(-2x/5)$. (Let $u = -x^2/5$ to make the function e^u.)

(8) $8(\sqrt{x}+x^3)^7\left(\dfrac{1}{2\sqrt{x}} + 3x^2\right)$. (Let $u = \sqrt{x} + x^3$ to make the function u^8.)

(9) $\dfrac{-1}{(\cos x)^2}(-\sin x) = \dfrac{\sin x}{\cos^2 x}$. (Let $u = \cos x$ to make the function $1/u$.)

(10) $-\sin\left(\dfrac{1}{x}\right)\left(\dfrac{-1}{x^2}\right) = \dfrac{\sin(1/x)}{x^2}$. (Let $u = 1/x$ to make the function $\cos u$.)

(11) $3^{\cos x}(\ln 3)(-\sin x) = -(\ln 3)3^{\cos x}(\sin x)$. (Let $u = \cos x$ to make the function 3^u.)

(12) $-\sin(5x)5 = -5\sin(5x)$. (Let $u = 5x$ to make the function $\cos u$.)

(13) $3^{\cos(5x)}(\ln 3)(-\sin(5x))5 = -5(\ln 3)3^{\cos(5x)}\sin(5x)$. (Let $u = \cos(5x)$ to make the function 3^u.)

(14) $-\sin(3^x)3^x(\ln 3) = -(\ln 3)(\sin(3^x))3^x$. (Let $u = 3^x$ to make the function $\cos u$.)

(15) $-\sin(\sqrt{x})\dfrac{1}{2\sqrt{x}} = \dfrac{-\sin(\sqrt{x})}{2\sqrt{x}}$. (Let $u = \sqrt{x}$ to make the function $\cos u$.)

(16) $\dfrac{1}{2\sqrt{\cos x}}(-\sin x) = \dfrac{-\sin x}{2\sqrt{\cos x}}$. (Let $u = \cos x$ to make the function $\sqrt{u}$.)

(17) $\dfrac{1}{4x+3}(4) = \dfrac{4}{4x+3}$. (Let $u = 4x + 3$ to make the function $\ln u$.)

(18) $-\dfrac{1}{(\ln x)^8}(4(\ln x)^3)\dfrac{1}{x} = \dfrac{-4(\ln x)^3}{x(\ln x)^8}$. (Let $u = (\ln x)^4$ to make the function $1/u$.)

(19) $(\sec^2(e^{-x}+4x^2))(-e^{-x}+8x)$. (Let $u = e^{-x} + 4x^2$ to make the function $\tan u$.)

(20) $(\sec^2((2x+1)^3+x))(3(2x+1)^2+1)(2) = 2(3(2x+1)^2+1)\sec^2((2x+1)^3+x)$. (Let $u = (2x+1)^3 + x$ to make the function $\tan u$.)

(21) $3\tan^2 x \sec^2 x$. (Let $u = \tan x$ to make the function u^3.)

(22) $(\cos(x^3-x))(3x^2-1)$. (Let $u = x^3 - x$ to make the function $\sin u$.)

(23) $4((x^2-x)^3+x+2)^3(3(x^2-x)^2(2x-1)+1)$. (Let $u = (x^2-x)^3 + x + 2$ to make the function u^4.)

(24) $\dfrac{-1}{(x^3+3^x)^2}(3x^2+3^x\ln 3)$. (Let $u = x^3 + 3^x$ to make the function $1/u$.)

(25) $-\sin(\ln(x^3+1))\dfrac{1}{x^3+1}(3x^2) = \dfrac{-3x^2\sin(\ln(x^3+1))}{x^3+1}$. (Let $u = \ln(x^3+1)$ to make the function $\cos u$.)

(26) $\dfrac{1}{\cos(x^3+1)}(-\sin(x^3+1))(3x^2) = \dfrac{-3x^2\sin(x^3+1)}{\cos(x^3+1)}$. (Let $u = \cos(x^3+1)$ to make the function $\ln u$.)

(27) $3(\ln(\cos x))^2\dfrac{1}{\cos x}(-\sin x) = \dfrac{-3(\ln(\cos x))^2\sin x}{\cos x}$. (Let $u = \ln(\cos x)$ to make the function u^3+1.)

(28) $e^{\sin(x^2)}(\cos(x^2))(2x) = 2x\cos(x^2)e^{\sin(x^2)}$. (Let $u = \sin(x^2)$ to make the function e^u.)

(29) $(\cos(e^{x^2}))e^{x^2}(2x) = 2xe^{x^2}\cos(e^{x^2})$. (Let $u = e^{x^2}$ to make the function $\sin u$.)

(30) $2(\sin(e^x))(\cos(e^x))e^x = 2e^x\sin(e^x)\cos(e^x)$. (Let $u = \sin(e^x)$ to make the function u^2.)