We will describe some applications of Zorn’s lemma to field extensions.

An algebraic closure of a field $K$ is an algebraic extension $L/K$ such that $L$ is algebraically closed. In [1, p. 544] there is a proof that every field admits an algebraic closure. The proof uses an iterative procedure starting with a polynomial ring in a very large number of variables and a maximal ideal of this ring that contains a certain proper ideal. Such a maximal ideal exists by Zorn’s lemma. A modification of this argument is given in the course handout on algebraic closures. For a different proof of the existence of algebraic closures, using Zorn’s lemma in a more direct fashion, see [2, pp. 259–260].

Now that we have the existence of algebraic closures, we will use Zorn’s lemma to prove any two algebraic closures of a field are isomorphic fields. The first step in the proof, which is where Zorn’s lemma is used, concerns the extension of a field homomorphism to a larger field.

**Theorem 1.** Let $L/K$ be an algebraic extension and let $C$ be an algebraically closed field. Any ring homomorphism $\varphi: K \to C$ can be extended to a homomorphism $L \to C$.

**Proof.** Let $S$ be the set of pairs $(F, f)$ where $F$ is an intermediate field between $K$ and $L$ and $f|_K = \varphi$. For instance $(K, \varphi) \in S$, so $S \neq \emptyset$.

Partially order $S$ by declaring $(F, f) \leq (F', f')$ if $F \subset F'$ and $f'|_F = f$. For a totally ordered subset $\{(F_\alpha, f_\alpha)\}_{\alpha \in A}$ in $S$, an upper bound can be produced as follows. Let $F = \cup_{\alpha \in A} F_\alpha$. Since the $F_\alpha$’s are totally ordered, it’s easy to check that $F$ is a field. Define $f: F \to C$ by $f(x) = f_\alpha(x)$ when $x \in F_\alpha$. If $x$ is also in $F_\beta$, we should check $f_\alpha(x) = f_\beta(x)$ so we know the definition of $f(x)$ is independent of the choice of $F_\alpha$ containing $x$. Since our subset of $S$ is totally ordered, either $(F_\alpha, f_\alpha) \leq (F_\beta, f_\beta)$ or $(F_\beta, f_\beta) \leq (F_\alpha, f_\alpha)$. In the first case, $f_\beta$ restricts to $f_\alpha$ on $F_\alpha$, so $f\beta(x) = f_\alpha(x)$. The argument in the second case is the same. For $x \in K$, we can view $x$ in any $F_\alpha$ and then $f(x) = f_\alpha(x) = \varphi(x)$ since $f_\alpha|_K = \varphi$, so $f|_K = \varphi$. To prove $f$ is a ring homomorphism, view any two elements of $F$ in a common $F_\alpha$ (total ordering) and use the properties of $f_\alpha$. Since $f|_{F_\alpha} = f_\alpha$, $(F, f)$ is an upper bound on all the $(F_\alpha, f_\alpha)$’s.

Now we can apply Zorn’s lemma: $S$ has a maximal element $(F, \sigma)$. That is, $F$ is a field between $K$ and $L$ with a homomorphism $\sigma: F \to C$ such that $\sigma|_K = \varphi$ and there is no extension of $\sigma$ to a homomorphism from a larger intermediate field to $C$. We will prove $F = L$, which means $\varphi$ extends up to $L$.

If $F \neq L$ then there is some $x \in L$ with $x \notin F$. Then $F(x)/F$ is a finite extension of degree greater than 1. We’re going to extend $\sigma$ to a homomorphism $F(x) \to C$. Let $g(X)$ be the minimal polynomial for $x$ in $F[X]$, so there is an $F$-isomorphism $F(x) \cong F[X]/(g(X))$. Applying $\sigma$ to the coefficients of $x$ gives a polynomial $g^\sigma(X) \in C[X]$. Since $C$ is algebraically closed, $g^\sigma(X)$ has a root in $C$, say $r$. Let $F[X] \to C$ by acting as $\sigma$ on $F$ and sending $X$ to $r$. This is a ring homomorphism that sends $g(X)$ to $g^\sigma(r) = 0$, so we get an induced homomorphism $F[X]/(g(X)) \to C$ acting as $\sigma$ on $F$ and sending $X$ to $r$. Composing this with the isomorphism $F(x) \cong F[X]/(g(X))$ from before gives us a
homomorphism \( \tau : F(x) \to C \) acting as \( \sigma \) on \( F \). Thus \( (F, \sigma) \leq (F(x), \tau) \). This is impossible by maximality of \( (F, \sigma) \), so \( F = L \). \( \square \)

The extension of \( \varphi \) to \( L \) is very far from unique.

**Corollary 2.** Let \( K \) be a field and \( i : K \to L \) be a ring homomorphism to another field \( L \) such that \( L/i(K) \) is an algebraic extension. For any ring homomorphism \( \varphi : K \to C \) into an algebraically closed field there is a ring homomorphism \( \sigma : L \to C \) such that \( \sigma \circ i = \varphi \).

**Proof.** Since \( i \) is a map of fields it is injective: \( K \) and \( i(K) \) are isomorphic fields using \( i \). Run through the above proof with the following change: \( S \) is the pairs \( (F, f) \) where \( F \) is a field between \( i \) and \( L \) and \( f \circ i = \varphi \) (rather than \( f|_K = \varphi \) as we used before). Define the partial ordering as before: \( (F, f) \leq (F', f') \) when \( F \subset F' \) and \( f'f = f \). It is left to the reader to check that \( S \) satisfies the assumptions of Zorn’s lemma and that a maximal element of \( S \) provides a solution to our problem. \( \square \)

Now we can establish the desired result about two algebraic closures.

**Corollary 3.** Any two algebraic closures of a field are isomorphic.

**Proof.** Let \( C_1 \) and \( C_2 \) be algebraic closures of the field \( K \). There are inclusion maps \( i_1 : K \to C_1 \) and \( i_2 : K \to C_2 \). Since \( C_2 \) is algebraically closed and \( C_1 \) is an algebraic extension of \( i_1(K) \), Corollary 2 tells us there is a ring homomorphism \( \sigma : C_1 \to C_2 \) such that \( \sigma \circ i_1 = i_2 \). The image \( \sigma(C_1) \) is an algebraically closed field that contains \( \sigma(i_1(K)) = i_2(K) \). Since \( C_2 \) is an algebraic extension of \( i_2(K) \), \( C_2/\sigma(C_1) \) is an algebraic extension of algebraically closed fields, so the extension has to be trivial: \( \sigma(C_1) = C_2 \). Thus \( \sigma \) is an isomorphism of \( C_1 \) and \( C_2 \). \( \square \)

There is a huge number of isomorphisms between two algebraically closed fields, so the construction in Corollary 3 is not at all canonical.

**Corollary 4.** Let \( K_1 \) and \( K_2 \) be isomorphic fields with respective algebraic closures \( C_1 \) and \( C_2 \). Any isomorphism \( K_1 \to K_2 \) extends to an isomorphism \( C_1 \to C_2 \). In particular, if \( K \) is a field with algebraic closure \( C \) then any field automorphism of \( K \) extends to a field automorphism of \( C \).

**Proof.** Let \( f : K_1 \to K_2 \) be a field isomorphism. Composing this with the inclusion \( i_2 : K_2 \to C_2 \) gives us a homomorphism \( i_2 \circ f : K_1 \to C_2 \). Apply Corollary 2 using \( L = C_1 \) and \( \varphi = i_2 \circ f \) to see there is a field homomorphism \( \sigma : C_1 \to C_2 \) such that \( \sigma \circ i_1 = \varphi = i_2 \circ f \), so we have a commutative diagram

\[
\begin{array}{ccc}
C_1 & \xrightarrow{\sigma} & C_2 \\
\uparrow{i_1} & & \uparrow{i_2} \\
K_1 & \xrightarrow{f} & K_2
\end{array}
\]

which shows \( \sigma(C_1) \) is an algebraically closed subfield of \( C_2 \) that contains \( \sigma(i_1(K_1)) = i_2(f(K_1)) = i_2(K_2) = K_2 \). As \( C_2 \) is an algebraic closure of \( K_2 \) we must have \( \sigma(C_1) = C_2 \), so \( \sigma \) is an isomorphism. \( \square \)

There is nothing canonical about the choice of \( \sigma \): the isomorphism \( f \) will generally have many extensions to an isomorphism of \( C_1 \) with \( C_2 \).

In addition to algebraic closures, certain fields have another, smaller, kind of closure called a real closure. Before we define a real closure, we have to define a real field. A field is called real (some use the label formally real) if \(-1\) is not a sum of squares in the field.
Obviously \( \mathbb{R} \) is an example of a real field. Any subfield of a real field is also real. More generally, any field that can be embedded in a real field is real. For example, the abstract field \( \mathbb{Q}(\theta) \) where \( \theta^2 = 2 \) is a real field since it is isomorphic to \( \mathbb{Q}(\sqrt{2}) \), which is real because it is a subfield of \( \mathbb{R} \). The fields \( \mathbb{C} \) and \( \mathbb{Q}(i) \) are obviously not real since \(-1\) is a square in these fields. Less obviously, \( \mathbb{Q}(\sqrt{-2}) \) is not real; although \(-1\) is not a square in \( \mathbb{Q}(\sqrt{-2}) \) it is a sum of two squares: \(-1 = (\sqrt{-2}/2)^2 + (\sqrt{-2}/2)^2\).

The concept of a real field is closely connected with the concept of an ordered field (a field on which one has a reasonable notion of positive and negative): the fields that admit an ordering — not necessarily just one — turn out to be precisely the real fields. For more information about real fields, see [3, Chap. XI].

A real closure of a field \( K \) is an extension field \( L \) that is algebraic, real, and admits no proper real algebraic extensions. In other words, it basically behaves for real fields like algebraic closures do for all fields (but a real closure is not algebraically closed, e.g., \( X^2 + 1 \) has no root in real fields). Since a real closure is a real field, and subfields of real fields are real, only real fields could have real closures. And indeed they all do, as we now prove.

**Theorem 5.** Every real field admits a real closure.

**Proof.** Let \( K \) be a real field. That is, \(-1\) is not a sum of squares in \( K \). A real closure of \( K \) would be a particular kind of algebraic extension, so we will work in a fixed algebraic closure \( C \supseteq K \). Since a real closure should arise as a maximal real algebraic extension, the way to use Zorn’s lemma should be obvious: take for \( S \) the set of all real algebraic extensions of \( K \) inside of \( C \). We know \( S \neq \emptyset \) since \( K \subseteq S \). Define a partial ordering on \( S \) by inclusion: \( F 
subseteq F' \) if \( F \subsetneq F' \). If \( \{F_\alpha\}_{\alpha \in A} \) is a totally ordered subset of \( S \) then the union \( F = \cup_{\alpha \in A} F_\alpha \) is a field (by the usual proof) that is inside of \( C \) and it contains each \( F_\alpha \). To see that \( F \) is real, assume otherwise: \(-1 = \sum_{i=1}^{n} c_i^2 \) where \( c_i \in F \). These finitely many \( c_i \)'s are in some common \( F_\alpha \) (since the \( F_\alpha \)'s are totally ordered), but then the equation violates the property of that \( F_\alpha \) being real. So in fact \(-1\) is not a sum of squares in \( F \), which means \( F \) is real and thus is an upper bound on \( \{F_\alpha\}_{\alpha \in A} \) in \( S \).

By Zorn’s lemma, \( S \) contains a maximal element. Denote one as \( R \). Then \( R \) is a real algebraic extension field of \( K \). Could \( R \) admit a proper real algebraic extension? Certainly not in \( C \), because \( R \) is maximal with respect to inclusion among real algebraic extensions of \( K \) in \( C \). If there was some extension \( R \hookrightarrow R' \) outside of \( C \) where \( R' \) is real and algebraic over \( R \), then by Theorem 1 we can embed \( R' \) into \( C \) by a map fixing \( R \). (\( C \) is an algebraic closure of \( R \).) In particular, the image of an embedding \( R' \hookrightarrow C \) fixing \( R \) pointwise will be a (necessarily) real field in \( C \) of degree greater than 1 over \( R \), but that contradicts maximality of \( R \) in \( S \). So \( R \) fits the definition of being a real closure of \( K \). \( \Box \)

As with algebraic closures, Zorn’s lemma can be used to prove any two real closures of a real field are isomorphic [3, p. 455], but the proof requires some preliminary work on “real roots” of polynomials that would be a bit of a diversion for us to review here. To appreciate the special role of real fields, where none of the polynomials \( \sum_{i=1}^{n} X_i^2 + 1 \) have zeros, observe that Zorn’s lemma implies that a field that does not contain a square root of \(-1\) admits a maximal algebraic extension that does not contain a square root of \(-1\), but two such maximal extensions need not be isomorphic. For example, the fields \( \mathbb{Q}(\sqrt{2}) \) and \( \mathbb{Q}(\sqrt{-2}) \) don’t contain a square root of \(-1\) and therefore each admits a maximal algebraic extension \( F/\mathbb{Q}(\sqrt{2}) \) and \( F'/\mathbb{Q}(\sqrt{-2}) \) not containing \( \sqrt{-1} \). Both \( F \) and \( F' \) are also maximal algebraic extensions of \( \mathbb{Q} \) not containing \( \sqrt{-1} \), but \( F \not\cong F' \) since \( X^2 - 2 \) has a root in \( F \).
but not in $F'$ (if $F'$ has a square root of 2 then its ratio with $\sqrt{-2} \in F'$ is a square root of $-1$ in $F'$, and that’s impossible).

We have focused our applications of Zorn’s lemma for fields to the case of algebraic extensions (specifically, algebraic closures and real closures). Zorn’s lemma is just as important in dealing with non-algebraic field extensions (such as $K(X, Y)$ where $X$ and $Y$ are indeterminantes). The central notion here is a transcendence basis, which is basically a maximal set of algebraically independent elements. It is a non-linear generalization of a vector space basis, and its existence in general uses Zorn’s lemma. See [3, Chap. VIII] or [5, pp. 357–373] for a detailed discussion of transcendence bases and [1, Sect. 14.9] for a survey on transcendence bases without proofs. As motivation for caring about transcendence bases, a careful study of them yields the following interesting result about algebraically closed fields, whose proof we omit.

**Theorem 6.** Two uncountable algebraically closed fields are isomorphic if and only if they have the same characteristic and cardinality.

For instance, any algebraically closed field of characteristic 0 whose cardinality is the same as that of $\mathbb{C}$ is isomorphic to $\mathbb{C}$ as an abstract field. (Examples of this situation really do occur, e.g., the algebraic closure of the $p$-adic numbers.)

Theorem 6 is false for countable algebraically closed fields. For instance, the algebraic closures of $\mathbb{Q}$ and $\mathbb{Q}(X)$ are both countable of characteristic 0, but they are not isomorphic fields since we can’t embed $\mathbb{Q}(X)$ inside an algebraic closure of $\mathbb{Q}$: the element $X$ is not algebraic over $\mathbb{Q}$.

Zorn’s lemma has other important uses in field theory. For example, Krull’s theorem about the extension of absolute values says any non-archimedean absolute value on a field $K$ admits an extension (generally not unique) to a non-archimedean absolute value on any larger field $L$. Much as the proof of Theorem 1 is reduced to the case of a simple field extension (that is, a field extension with a single generator) using Zorn’s lemma, the proof of Krull’s theorem is reduced by Zorn’s lemma to the case of a simple extension $L = K(t)$, and one can then treat the cases when $t$ is algebraic over $K$ or transcendental over $K$ separately [6, pp. 36–39]. For an alternate proof of Krull’s theorem using Zorn’s lemma, see [4, pp. 107–108].

**References**