THE INFINITUDE OF THE PRIMES

KEITH CONRAD

1. Introduction

The sequence of prime numbers

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, \dots, 1873, 1877, 1879, 1889, 1901, \dots$$

never ends. This fact has many different proofs. We'll discuss three of them, due to Euclid, Euler, and Erdős and some of their consequences. More proofs are in [10, Chap. 1].

2. Euclid's proof

The standard proof of the infinitude of the primes is attributed to Euclid and uses the fact that all integers greater than 1 have a prime factor.

Lemma 2.1. *Every integer greater than* 1 *has a prime factor.*

Proof. We argue by (strong) induction that each integer $n > 1$ has a prime factor. For the base case $n = 2$, 2 is prime and is a factor of itself.

Now assume $n > 2$ all integers greater than 1 and less than n have a prime factor. To show n has a prime factor, we take cases.

Case 1: n is prime.

Since n is a factor of itself, n has a prime factor when n is prime.

Case 2: n is not prime.

Since n is not prime, it has a factorization $n = ab$ where $1 < a, b < n$. Then by the strong inductive hypothesis, a has a prime factor, say p. Since $p \mid a$ and $a \mid n$, also $p \mid n$ and thus n has prime factor p. □

Theorem 2.2. *There are infinitely many primes.*

Proof. (Euclid) To show there are infinitely many primes, we'll show that every finite list of primes is missing a prime number, so the list of all primes can't be finite.

To begin, there are prime numbers such as 2. Suppose $p_1, \dots, p_r$ is a finite list of prime numbers. We want to show this is not the full list of the primes. Consider the number

$$N = p_1 \cdots p_r + 1.$$

Since $N > 1$, it has a prime factor p by Lemma 2.1. The prime p can't be any of $p_1, \dots, p_r$ since N has remainder 1 when divided by each p_i. Therefore p is a prime not on our list, so the set of primes can't be finite. □

Some people misunderstand this proof to be saying that if $p_1, \dots, p_r$ are prime then $p_1 \cdots p_r + 1$ is prime. That is not generally true. It starts out looking correct: $2 + 1$ is prime, $2 \cdot 3 + 1 = 7$ is prime, $2 \cdot 3 \cdot 5 + 1 = 31$ is prime, $2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211$ is prime, and $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1 = 2311$ is prime, but

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031 = 59 \cdot 509$$

is *not* prime. (More simply, $2 \cdot 7 + 1 = 15$ and $3 \cdot 5 + 1 = 16$ are not prime.) Euclid's proof tells us that we can always find a prime outside of a finite list of primes $p_1, \dots, p_r$ by using a prime factor of $p_1 \cdots p_r + 1$, not by using $p_1 \cdots p_r + 1$ itself.

Let's try to use Euclid's proof as a method of finding new primes, by taking the *smallest* prime factor of $p_1 \cdots p_r + 1$ at each step as a new prime to add to our list:

- 2 is prime and $2 + 1 = 3$ is prime,
- $2 \cdot 3 + 1 = 7$ is prime,
- $2 \cdot 3 \cdot 7 + 1 = 43$ is prime,
- $2 \cdot 3 \cdot 7 \cdot 43 + 1 = 1807 = 13 \cdot 139$,
- $2 \cdot 3 \cdot 7 \cdot 43 \cdot 13 + 1 = 23479 = 53 \cdot 443$,
- $2 \cdot 3 \cdot 7 \cdot 43 \cdot 13 \cdot 53 + 1 = 1244335 = 5 \cdot 248867$.

So far this list of primes in the order they appear is $2, 3, 7, 43, 13, 53, 5$. This way of creating new primes was introduced by Mullin [8] in 1963 and is called the Euclid–Mullin sequence. Further terms in this sequence are on the OEIS webpage https://oeis.org/A000945: 11 is the 12th term, 17 is the 13th term, 19 is the 36th term, and 23 is the 25th term. Mullin asked if every prime number actually appears as some term in this sequence; this is an unsolved problem.

Mullin also considered a similar algorithm to generate new primes by using the *largest* prime factor of $p_1 \cdots p_r + 1$ at each step. This "second Euclid–Mullin sequence" starts off as $2, 3, 7, 43, 139, 50207, 340999$. If this list is continued, some primes definitely never show up in it, such as all the primes from 5 to 47 [5]. In fact, infinitely many primes are missing from this sequence [2], [9].

For positive x, the number of primes less than or equal to x is written as $\pi(x)$. For example, $\pi(10) = |\{2, 3, 5, 7\}| = 4$ and $\pi(12.7) = |\{2, 3, 5, 7, 11\}| = 5$. Because there are infinitely many primes, $\pi(x) \to \infty$ as $x \to \infty$. Euclid's proof gives us the following crude lower bound on the number of primes up to x.

Corollary 2.3. *For $x \geq 2$, $\pi(x) > \log(\log x)$.*

Proof. If $p_1, \dots, p_n$ are the first n primes, then Euclid's proof tells us that a prime factor of $p_1 \cdots p_n + 1$ is a new prime, so $p_{n+1} \leq p_1 \cdots p_n + 1$. Using this inequality, we can show by induction that $p_n \leq 2^{2^{n-1}}$ for all $n \geq 1$: it's true when $n = 1$ since $2 \leq 2^{2^{1-1}}$, and if that upper bound on primes is true for $p_1, \dots, p_k$ then

$$p_{k+1} \leq p_1 p_2 \cdots p_k + 1 \leq 2 \cdot 2^2 \cdots 2^{2^{k-1}} + 1 = 2^{1+2+\cdots+2^{k-1}} + 1 = 2^{2^k - 1} + 1,$$

and that last number is at most 2^{2^k} since $\frac{1}{2}x + 1 \leq x$ for $x \geq 2$ (take $x = 2^{2^k}$).

That $p_n \leq 2^{2^{n-1}}$ for all $n \geq 1$ means the number of primes up to $2^{2^{n-1}}$ is at least n, so $\pi(2^{2^{n-1}}) \geq n$ for $n \geq 1$. For $x \geq 2$, choose the integer $n \geq 1$ such that $2^{n-1} \leq \log_2 x < 2^n$. Then $x \geq 2^{2^{n-1}}$, so $\pi(x) \geq \pi(2^{2^{n-1}}) \geq n > \log_2(\log_2 x)$. Since $0 < \log 2 < 1$, $\log_2(y) = (\log y)/(\log 2) > \log y$ for $y > 0$, so $\log_2(\log_2 x) > \log(\log x)$. □

This lower bound on $\pi(x)$ is extremely weak because the true order of magnitude of $\pi(x)$ is $x/\log x$: the ratio $\pi(x)/(x/\log x)$ tends to 1 as $x \to \infty$. That is called the Prime Number Theorem.

3. Euler's proof

In 1737, Euler [7, Theorem 7] found a proof of the infinitude of the primes that explains it by the divergence of the harmonic series. This unexpected link between a property of prime numbers and calculus (infinite series) could be considered the start of the subject of *analytic number theory*, which studies properties of $\mathbf{Z}$ using the tools of real and complex analysis. While Euclid's proof used the fact that each integer greater than 1 has a prime factor, Euler's proof will rely on unique factorization in $\mathbf{Z}^+$.

Theorem 3.1. *There are infinitely many primes.*

Proof. (Euler) For a prime p, the ratio $1/(1-1/p)$ can be expanded into a geometric series:

$$\frac{1}{1-1/p} = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \cdots$$

We will now multiply together these ratios for different primes. To see what can be obtained, let's look at the product of these terms for the primes 2, 3, and 5: $\frac{1}{1-1/2}\frac{1}{1-1/3}\frac{1}{1-1/5}$ equals

$$\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\right)\left(1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots\right),$$

and if we multiply together one term from each series (this is the distributive law for multiplying infinite series, and it is valid when multiplying convergent series of positive numbers) we will get unit fractions $1/n$ where n is a product of powers of 2, 3, and 5:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \frac{1}{16} + \frac{1}{18} + \frac{1}{20} + \frac{1}{24} + \cdots \tag{3.1}$$

This looks like the harmonic series, but it is missing terms $1/n$ where n has a prime factor greater than 5, such as terms $1/7$, $1/11$, and $1/14$. If we multiply by additional factors $1/(1-1/p)$ for more primes p we'll introduce into (3.1) new terms $1/n$ where n has prime factors involving the new primes and the ones we already used (2, 3, and 5). Because of unique factorization in $\mathbf{Z}^+$, each term $1/n$ that appears will do so just once. Therefore if we multiply together $1/(1-1/p)$ for *all* the primes,[1] (3.1) turns into the sum of $1/n$ over all positive integers n:

$$\prod_p \frac{1}{1-1/p} = \sum_{n\geq 1} \frac{1}{n}, \tag{3.2}$$

where the product on the left (denoted Π, like Σ for sum) runs over all prime numbers.

Since the harmonic series diverges, (3.2) tells us that the left side can *not* be a product of finitely many terms. Therefore there are infinitely many terms in the product, so there are infinitely many primes. □

By taking logarithms of a product over primes, Euler showed something new about primes when we sum their reciprocals:

Corollary 3.2. *The infinite series $\sum_p 1/p$, where p runs over the primes, diverges.*

Proof. For $N \geq 2$, multiply out

$$\prod_{p\leq N} \frac{1}{1-1/p} = \prod_{p\leq N}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \dots\right),$$

where the product is over primes p up to N, by multiplying together one term from each of the geometric series on the right. We obtain unit fractions $1/n$ that include every integer

[1] An infinite product, like an infinite series, is a limit of partial products: $a_1a_2a_3\cdots = \lim_{k\to\infty} a_1a_2\cdots a_k$.

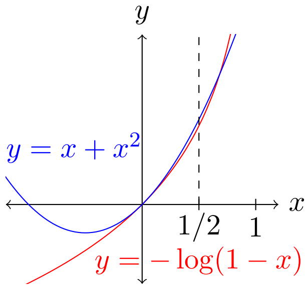
$n \le N$, since the prime factors of such n are all at most N. We'll also get unit fractions $1/n$ for other n. Therefore

$$\sum_{n \le N} \frac{1}{n} < \prod_{p \le N} \frac{1}{1-1/p} \tag{3.3}$$

As $N \to \infty$ the left side of (3.3) tends to ∞ as N does, so the right side of (3.3) tends to ∞ too. That isn't really telling us anything new, since (3.2) already says $\prod_p \frac{1}{1-1/p} = \infty$. We can get something new from (3.3) by taking logarithms of both sides before letting N tend to ∞: logarithms turn products into sums and $\log x$ is increasing, so (3.3) implies

$$\log\left(\sum_{n \le N} \frac{1}{n}\right) \le \sum_{p \le N} \log \frac{1}{1-1/p} \tag{3.4}$$

On the right, $\log(1/(1-1/p)) = -\log(1-1/p)$. Recall for $|x| < 1$ that the power series for $-\log(1-x)$ is $\sum_{n \ge 1} x^n/n = x + x^2/2 + x^3/3 + \cdots$, so for small x we have $-\log(1-x) \approx x + x^2/2$. We need a definite inequality here, not just "$\approx$", and for that increase $x^2/2$ up to x^2: for $0 < x \le 1/2$, verify by calculus that $-\log(1-x) < x + x^2$ (see the graph below).



Using $x = 1/p$ for prime p, so $x \le 1/2$, we have $-\log(1-1/p) < 1/p + 1/p^2$. Thus

$$\sum_{p \le N} -\log(1-1/p) < \sum_{p \le N} \left(\frac{1}{p} + \frac{1}{p^2}\right).$$

Combining this with (3.4),

$$\log\left(\sum_{n \le N} \frac{1}{n}\right) < \sum_{p \le N} \frac{1}{p} + \sum_{p \le N} \frac{1}{p^2}. \tag{3.5}$$

Now let $N \to \infty$: the left side tends to ∞ since $\sum_{n \le N} 1/n \to \infty$ and on the right side $\sum_{p \le N} 1/p^2$ has a finite limit as $N \to \infty$ since $\sum_{n \ge 1} 1/n^2$ converges. Therefore (3.5) implies $\sum_{p \le N} 1/p \to \infty$ as $N \to \infty$, so $\sum_p 1/p = \infty$. □

Since $\sum_{n \ge 1} 1/n = \infty$ and $\sum_p 1/p = \infty$ while $\sum_{n \ge 1} 1/n^2 < \infty$, the primes are "more dense" among the positive integers than the squares.

We can derive Corollary 2.3 from Euler's approach to the infinitude of the primes.

Corollary 3.3. *For $x \ge 2$, $\pi(x) > \log(\log x)$.*

Proof. In inequality (3.3), on the right side $1/(1-1/p) \leq 2$, so $\prod_{p\leq N} 1/(1-1/p) \leq 2^{\pi(N)}$. On the left side, using a Riemann sum approximation to $\int_1^{N+1} dx/x$ with rectangles of width 1, we have

$$\sum_{n\leq N} \frac{1}{n} > \int_1^{N+1} \frac{dx}{x} = \log(N+1).$$

Therefore $\log(N+1) < 2^{\pi(N)}$ when $N \geq 2$. For $x \geq 2$, letting $N \leq x < N+1$ for an integer N, we get

$$\log x < \log(N+1) < 2^{\pi(N)} = 2^{\pi(x)} < e^{\pi(x)},$$

so $\pi(x) > \log(\log x)$. □

Although $\sum_p 1/p = \infty$, the divergence is *extremely* slow. For example, $\sum_{p<10^8} 1/p \approx 3.174975$. The partial sums $\sum_{p\leq N} 1/p$ diverge at the rate $\log(\log N)$, which Euler knew: the last line of his paper where he proved infinitude of the primes [7] says

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \text{etc.} = l.\,l\,\infty,$$

where l was Euler's notation for natural logarithms.

In a MathOverflow post about the importance of rigor[2], the top answer by Richard Borcherds says "I once got a letter from someone who had overwhelming numerical evidence that the sum of the reciprocals of primes is slightly bigger than 3 (he may have conjectured the limit was π). The sum is in fact infinite, but diverges so slowly (like log log n) that one gets no hint of this by computation." We saw $\sum_{p<10^8} 1/p$ is slightly above 3. Since the solution to $\log(\log x) = 4$ is a 24-digit number, a calculator or computer will never give convincing evidence of the divergence of $\sum_p 1/p$.

Corollary 3.2 was very important historically in number theory: it suggests that a way to prove a set S of primes is infinite is to show the sum $\sum_{p\in S} 1/p$ is ∞. In fact, 100 years after Euler's work Dirichlet used this method in 1837 to prove certain arithmetic progressions contain infinitely many primes.

Theorem 3.4 (Dirichlet). *If a and m are relatively prime positive integers, there are infinitely many primes in the arithmetic progression $\{a + mn : n \geq 0\}$. Equivalently, there are infinitely many primes p such that $p \equiv a \bmod m$.*

For example, the set of primes p with units digit 7, namely $p \equiv 7 \bmod 10$, is infinite. It starts out as

$$17, 37, 47, 57, 67, 97, 107, 127, 137, 157, 167, 197, 227, 257, 277, \ldots$$

The basic idea in Dirichlet's proof is to show the series $\sum_{p\equiv a \bmod m} 1/p$ over primes p is infinite when $(a, m) = 1$, so the set of primes $p \equiv a \bmod m$ is infinite. Some special cases of Dirichlet's theorem (such as infinitude of primes p where $p \equiv 3 \bmod 4$) can be proved by elementary techniques, but I am unaware of any elementary method that shows the list of primes $p \equiv 7 \bmod 10$ is infinite.

A famous unsolved problem in number theory is the infinitude of *twin primes*, which are prime pairs that differ by 2, such as 3 and 5 or 29 and 31. If T is the set of primes p such that $p+2$ is also prime, might $\sum_{p\in T} 1/p$ be infinite? That would imply there are infinitely many twin primes. Unfortunately, $\sum_{p\in T} 1/p$ is known to converge (which doesn't contradict the

[2] See https://mathoverflow.net/questions/37610.

possibility that T is infinite: infinite series can converge). The convergence of $\sum_{p\in T} 1/p$ was proved by Brun [3] in 1919. The convergent series $B_2 := \sum_{p\in T}(1/p + 1/(p+2))$ is called Brun's constant. It is known to be less than 2.4 and is expected to be approximately 1.902. Estimating B_2 is a difficult problem, and work on this in 1994 led to the discovery of a bug in an Intel Pentium chip.

4. Erdős's proof

In a footnote to a short paper [6] on divergence of the series $\sum_p 1/p$ over the primes, Erdős gave the following combinatorial proof that there are infinitely many primes.

Theorem 4.1. *There are infinitely many primes.*

Proof. Suppose there are finitely many primes $p_1, p_2, \ldots, p_r$. Each positive integer up to N can be written uniquely as a^2b where a is a positive integer and b is squarefree. Since $a \leq \sqrt{N}$, the number of choices for a is at most $\sqrt{N}$. Since b is a product of distinct primes and there are r primes (including an empty product for $b = 1$), the number of choices of b is 2^r. Therefore the number of possibilities for a^2b is at most $\sqrt{N}2^r$. There are N positive integers up to N, so $N \leq \sqrt{N}2^r$. By algebra, $N \leq 2^{2r}$. This is a contradiction for large enough N. □

Corollary 4.2. *For every integer $N \geq 2$, $\pi(N) > (\log N)/(2\log 2)$.*

This lower bound on $\pi(N)$ is an improvement on the lower bound $\pi(x) > \log(\log x)$ that we saw earlier from the proofs of Euclid and Euler.

Proof. Let $r = \pi(N)$. Running through the proof of Theorem 4.1 with $p_1, \ldots, p_r$ being the set of primes up to N (not the set of all primes), by writing positive integers as a^2b for squarefree b we find that the number of integers up to N is at most $\sqrt{N}2^r$. Therefore $N \leq \sqrt{N}2^r = \sqrt{N}2^{\pi(N)}$, so $\pi(N) \geq (\log_2 N)/2 = (\log N)/(2\log 2)$. □

We can improve on the lower bound in Corollary 4.2 by using a result called Bertrand's postulate: for each integer $n \geq 2$, there is a prime number lying strictly between n and $2n$.[3]

Theorem 4.3. *Bertrand's postulate implies $\pi(x) > \log x$ for all $x \geq 3$.*

The inequality $\pi(x) > \log x$ is false for $e < x < 3$, where $\pi(x) = 1 < \log x$.

Proof. For $x \geq 8$, let $2^k \leq x < 2^{k+1}$ for an integer k. The $k-1$ open intervals $(2,4), (4,8)$, $(8,16), \ldots, (2^{k-1}, 2^k)$ each contain a prime number and all of them are all less than x, so $\pi(x) \geq k-1$. This lower bound on $\pi(x)$ does not include the prime 2 and for the interval $(4,8)$ we counted just one prime in it but there are two primes in it. Therefore

$$\pi(x) \geq (k-1) + 2 = k + 1 > \log_2 x > \log x$$

for $x \geq 8$. For $3 \leq x < 5$, $\log x < 2 \leq \pi(x)$ and for $5 \leq x \leq 8$, $\log x < 3 \leq \pi(x)$. □

[3] The label "Bertrand's postulate" is used for historical reasons and would be better described as Bertrand's conjecture. It was proved by Chebyshev [4, Sect. VI] after being conjectured by Bertrand [1, p. 129] in order to prove the symmetric group S_m has no subgroup with index between 2 and m when $m \geq 3$.

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