# NEGATION AND INVERSION OF CONTINUED FRACTIONS

KEITH CONRAD

Gosper developed general algorithms for adding, subtracting, multiplying, and dividing continued fractions. Here we record the formulas for negation and inversion of continued fraction representations. Inversion will have 10 (!) cases.

Let $\alpha = [a_1, a_2, a_3, \dots]$ be a continued fraction, with $a_1 \in \mathbf{Z}$ and $a_n \in \mathbf{Z}^+$ for $n \geq 2$. If $\alpha$ is rational then we simply omit $a_n$ beyond some point (or formally take such $a_n$ to be 0).

Negation.

$$-\alpha = \begin{cases} [-a_1 - 1, 1, a_2 - 1, a_3, a_4, \dots], & \text{if } a_2 \geq 2, \\ [-a_1 - 1, a_3 + 1, a_4, \dots], & \text{if } a_2 = 1. \end{cases}$$

The case $\alpha = [a_1]$, where formally $a_n = 0$ for $n \geq 2$, is covered by the last case since $[-a_1 - 1, 1] = (-a_1 - 1) + 1/1 = -a_1 = -\alpha$.

Inversion.

We collect the 10 cases for $1/\alpha$ into the table below. The simplest cases are the first two, corresponding to $\alpha \geq 1$ and $0 < \alpha < 1$. Handling $\alpha < 0$ requires 8 cases.

| Case | Condition | $1/\alpha$ |
|---|---|---|
| 1 | $a_1 \geq 1$ | $[0, a_1, a_2, a_3, \dots]$ |
| $1'$ | $a_1 = 0$ | $[a_2, a_3, \dots]$ |
| 2 | $a_1 \leq -3$ and $a_2 \geq 2$ | $[-1, 1, \lvert a_1\rvert - 2, 1, a_2 - 1, a_3, a_4 \dots]$ |
| $2'$ | $a_1 \leq -3$ and $a_2 = 1$ | $[-1, 1, \lvert a_1\rvert - 2, 1 + a_3, a_4, \dots]$ |
| 3 | $a_1 = -2$ and $a_2 \geq 2$ | $[-1, 2, a_2 - 1, a_3, a_4, \dots]$ |
| $3'$ | $a_1 = -2$ and $a_2 = 1$ | $[-1, 2 + a_3, a_4, \dots]$ |
| 4 | $a_1 = -1$ and $a_2 \geq 3$ | $[-2, 1, a_2 - 2, a_3, a_4, \dots]$ |
| $4'$ | $a_1 = -1$ and $a_2 = 2$ | $[-2, 1 + a_3, a_4, \dots]$ |
| 5 | $a_1 = -1$, $a_2 = 1$, and $a_4 \geq 2$ | $[-(2 + a_3), 1, a_4 - 1, a_5, \dots]$ |
| $5'$ | $a_1 = -1$, $a_2 = 1$, and $a_4 = 1$ | $[-(2 + a_3), 1 + a_5, a_6, \dots]$ |

Comments about the different cases:

2) This is valid for $[a_1, a_2]$ if $a_1 \leq -3$ and $a_2 \geq 2$.

$2'$) This is valid if $\alpha = [a_1, 1]$ and $\alpha = [a_1]$ when $a_1 \leq -3$.

3) This is valid for $\alpha = [-2, 2]$.

$3'$) This is valid for $\alpha = [-2, 1]$ and $\alpha = [-2]$.

4) This is valid for $\alpha = [-1, a_2]$ when $a_2 \geq 3$.

$4'$) This is *not* valid for $\alpha = [-1, 2]$. See case $5'$ for that.

5) This is valid for $\alpha = [-1, 1, a_3, a_4]$ if $a_4 \geq 2$.

$5'$) This is valid for $\alpha = [-1, 1, a_3]$ for $a_3 \geq 1$ using $a_n = 0$ for $n \geq 4$, including as a special case $\alpha = [-1, 1, a_3, 1]$ rewritten as $[-1, 1, a_3 + 1]$, and also $\alpha = [-1, 1, 1] = [-1, 2]$.

**Example**. If $\alpha = [-5, 9, 1]$ then by Case 2, $1/\alpha = [-1, 1, 3, 1, 8, 1] = [-1, 1, 3, 1, 9]$.

The only choice for $\alpha$ that we did not cover is $\alpha = [-1, 1]$, which is okay since that is 0.