

NOETHERIAN RINGS

KEITH CONRAD

1. INTRODUCTION

In a PID, every ideal has a single generator. In a ring that is not a PID, there may not be a bound on the number of generators of all the ideals. For example, in the polynomial ring $\mathbf{Q}[X, Y]$, the ideal (X, Y) has a generating set of size 2 but not one of size 1 (it is not principal), and the ideal $(X^n, X^{n-1}Y, \dots, XY^{n-1}, Y^n)$ in $\mathbf{Q}[X, Y]$ has a generating set of size $n + 1$ but not one of size n . Despite these variations, there is an important finiteness property of the ring $\mathbf{Q}[X, Y]$: all of its ideals are finitely generated.

Definition 1.1. A commutative ring R is called *Noetherian* if each ideal in R is finitely generated.

This name honors Emmy Noether, who in her landmark paper [4] in 1921 proved properties of such rings by conceptual methods instead of by computations in particular examples (like polynomial rings over a field). She did not write about “Noetherian rings”, but instead said such rings satisfy “the finiteness condition” (die Endlichkeitsbedingung).

2. EXAMPLES

A simple (and boring) example of a Noetherian ring is a field. A more general class of examples are PIDs, since all of their ideals are singly generated. Noetherian rings can be regarded as a good generalization of PIDs: the property of all ideals being singly generated is often not preserved under common ring-theoretic constructions (*e.g.*, \mathbf{Z} is a PID but $\mathbf{Z}[X]$ is not), but the property of all ideals being finitely generated does remain valid under many constructions of new rings from old rings. For example, we will see below that every quadratic ring $\mathbf{Z}[\sqrt{d}]$ is Noetherian, even though many of these rings are not PIDs.

The standard example of a non-Noetherian ring is a polynomial ring $K[X_1, X_2, \dots]$ in infinitely many variables over a field K . So that you don’t think non-Noetherian rings must be “really huge”, there is a non-Noetherian ring contained in $\mathbf{Q}[X]$: the ring of integral-valued polynomials

$$\text{Int}(\mathbf{Z}) = \{f \in \mathbf{Q}[X] : f(\mathbf{Z}) \subset \mathbf{Z}\}$$

is not Noetherian. This ring is bigger than $\mathbf{Z}[X]$, *e.g.*, $\binom{X}{2} = \frac{X(X-1)}{2}$ is in $\text{Int}(\mathbf{Z}) - \mathbf{Z}[X]$, as is $\binom{X}{n} = \frac{X(X-1)\cdots(X-n+1)}{n!}$ for all $n \geq 2$.

3. PROPERTIES OF NOETHERIAN RINGS

Theorem 3.1. *The following conditions on a commutative ring R are equivalent:*

- (1) *R is Noetherian: all ideals of R are finitely generated.*
- (2) *each infinite increasing sequence of ideals $I_1 \subset I_2 \subset I_3 \subset \dots$ in R eventually stabilizes: $I_k = I_{k+1}$ for all large k .¹*

¹The notation \subset only means containment, not strict containment.

(3) *Every nonempty collection S of ideals of R contains a maximal element with respect to inclusion: there's an ideal in S not strictly contained in another ideal in S .*

The first theorem in Noether's paper [4, p. 30] is that (1) \Rightarrow (2), and she called this the "theorem of the finite chain" (Satz von der endlichen Kette). The standard label for property (2) is the *ascending chain condition* or ACC. Immediately after proving (1) \Rightarrow (2), she observed that (2) \Rightarrow (1) and therefore that (2) could be used as a definition of her "finiteness condition" in place of (1).

Proof. (1) \Rightarrow (2): If $I_1 \subset I_2 \subset \cdots$ is an increasing sequence of ideals, let $I = \bigcup_{n \geq 1} I_n$. This is an ideal since each pair of elements in I lies in a common I_n , by the increasing condition, so I is closed under addition and multiplication by elements of R . By (1), I is finitely generated. Using the increasing condition again, each finite subset of I lies in a common I_n , so a finite generating set of I is in some I_m . Thus $I \subset I_m$, and of course also $I_m \subset I$, so $I = I_m$. Then for all $n \geq m$, $I_m \subset I_n \subset I = I_m$, so $I_n = I_m$.

(2) \Rightarrow (1): We prove the contrapositive. Suppose (1) is false, so R has an ideal I that is not finitely generated. Pick $r_1 \in I$. Since I is not finitely generated, $I \neq (r_1)$, so there is an $r_2 \in I - (r_1)$. Since $I \neq (r_1, r_2)$, there is an $r_3 \in I - (r_1, r_2)$. Proceed in a similar way to pick r_n in I for all $n \geq 1$ by making $r_n \in I - (r_1, \dots, r_{n-1})$ for $n \geq 2$. Then we have an increasing sequence of ideals $(r_1) \subset (r_1, r_2) \subset \cdots \subset (r_1, \dots, r_n) \subset \cdots$ in R where each ideal is strictly contained in the next one, so (2) is false.

(2) \Rightarrow (3): We will prove the contrapositive. If (3) is false then there is a nonempty collection S of ideals in R containing no maximal member with respect to inclusion. Therefore if we start with an ideal I_1 in R , we can recursively find ideals I_2, I_3, \dots such that I_n strictly contains I_{n-1} for all $n \geq 2$. (If there were no ideal in S strictly containing I_{n-1} , then I_{n-1} would be a maximal element of S , which doesn't exist.)

(3) \Rightarrow (1): Let I be an ideal in R . To prove I is finitely generated, let S be the set of all finitely generated ideals contained in I . By (3), there is an $\tilde{I} \in S$ that's contained in no other element of S , so \tilde{I} is a finitely generated ideal in I and no other finitely generated ideal of R contains \tilde{I} . We will show $\tilde{I} = I$ by contradiction, which would prove I is finitely generated. If $\tilde{I} \neq I$, pick $a \in I - \tilde{I}$. Since \tilde{I} is finitely generated, also $\tilde{I} + Ra$ is finitely generated, so $\tilde{I} + Ra \in S$. However, $\tilde{I} + Ra$ strictly contains \tilde{I} , which contradicts maximality of \tilde{I} as a member of S . Thus $\tilde{I} = I$. \square

The following two theorems put the second condition of Theorem 3.1 to use.

Theorem 3.2. *Let R be a Noetherian ring. Each surjective ring homomorphism $R \rightarrow R$ is injective, and thus is an isomorphism.*

Proof. Let $\varphi: R \rightarrow R$ be a surjective ring homomorphism. For the n th iterate φ^n (the n -fold composition of φ with itself), let $K_n = \ker(\varphi^n)$. This is an ideal in R and these ideals form an increasing chain:

$$K_1 \subset K_2 \subset K_3 \subset \cdots$$

since $r \in K_n \Rightarrow \varphi^n(r) = 0 \Rightarrow \varphi^{n+1}(r) = \varphi(\varphi^n(r)) = \varphi(0) = 0$, so $r \in K_{n+1}$. Since R is a Noetherian ring, $K_n = K_{n+1}$ for some n . Pick $r \in \ker \varphi$, so $\varphi(r) = 0$. The map φ^n is surjective since φ is surjective, so $r = \varphi^n(r')$ for some $r' \in R$. Thus $0 = \varphi(r) = \varphi(\varphi^n(r')) = \varphi^{n+1}(r')$. Therefore $r' \in \ker(\varphi^{n+1}) = \ker(\varphi^n)$, so $r = \varphi^n(r') = 0$. That shows $\ker \varphi = \{0\}$, so φ is injective. \square

There is no analogue of Theorem 3.2 for injective ring homomorphisms. For example, $\mathbf{R}[X]$ is a Noetherian ring since it's a PID and the substitution homomorphism $f(X) \mapsto f(X^2)$ on $\mathbf{R}[X]$ is an injective ring homomorphism that is not surjective.

Theorem 3.3. *If R is a Noetherian integral domain that is not a field, then every nonzero nonunit in R can be factored into irreducibles.*

We assume R is not a field because irreducible factorizations don't have a meaning for units, so we want R to contain some nonzero elements that aren't units.

Proof. This will be a proof by contradiction.

Suppose there is an element a in R that is not 0 or a unit and has no irreducible factorization. We will find another nonzero nonunit $b \in R$ that does not admit a factorization into irreducibles and such that there is a strict inclusion of ideals $(a) \subset (b)$.

Since a is not irreducible and it is not 0 or a unit, there is a factorization $a = bc$ where b and c are nonunits (and obviously they are not 0 either). If both b and c have an irreducible factorization then so does a (just multiply together irreducible factorizations for b and c), so at least one of b or c has no irreducible factorization. Without loss of generality, say b has no irreducible factorization. Then since c is not a unit, the inclusion $(a) \subset (b)$ is strict.

Rewriting a as a_1 and b as a_2 , we have a strict containment of ideals

$$(a_1) \subset (a_2)$$

where a_2 is a nonzero nonunit with no irreducible factorization. Using a_2 in the role of a_1 in the previous paragraph, there is a strict inclusion of ideals

$$(a_2) \subset (a_3)$$

for some nonzero nonunit a_3 that has no irreducible factorization. This argument can be repeated and leads to an infinite increasing chain of (principal) ideals

$$(3.1) \quad (a_1) \subset (a_2) \subset (a_3) \subset \cdots$$

where all inclusions are strict. This is impossible in a Noetherian ring, so we have a contradiction. Therefore nonzero nonunits without an irreducible factorization do not exist in R : all nonzero nonunits in R have an irreducible factorization. \square

This theorem is *not* saying a Noetherian integral domain has unique factorization: just because elements have irreducible factorizations doesn't mean those are unique (up to the order of multiplication and multiplication of terms by units). Many Noetherian integral domains do not have unique factorization.

We now show that some basic operations on rings preserve the property of being Noetherian.

Theorem 3.4. *If R is a Noetherian ring then so is R/I for each ideal I in R .*

Proof. Every ideal in R/I has the form J/I for an ideal J of R such that $I \subset J \subset R$. Since R is a Noetherian ring, J is a finitely generated ideal in R , and that finite generating set for J reduces to a generating set for J/I as an ideal of R/I . \square

To create more examples of Noetherian rings we can use the following very important theorem.

Theorem 3.5 (Hilbert Basis Theorem). *If R is a Noetherian ring then so is $R[X]$.*

The reason for the name “Basis Theorem” is historical. A generating set for an ideal used to be called a “basis” for the ideal (even though it’s not linearly independent). The theorem says if each ideal in R has a “finite basis” then the same is true of ideals in $R[X]$.

Proof. The theorem is clear if $R = 0$, so assume $R \neq \{0\}$. To prove each ideal I in $R[X]$ is finitely generated, we assume I is not finitely generated and will get a contradiction.

We have $I \neq (0)$. Define a sequence of polynomials f_1, f_2, \dots in I as follows.

- (1) Pick f_1 to be an element of $I - (0)$ with minimal degree. (It is not unique.)
- (2) Since $I \neq (f_1)$, as I is not finitely generated, pick f_2 in $I - (f_1)$ with minimal degree. Note $\deg f_1 \leq \deg f_2$ by the minimality condition on $\deg f_1$.
- (3) For $k \geq 2$, if we have defined f_1, \dots, f_k in I then $I \neq (f_1, \dots, f_k)$ since I is not finitely generated, so we may pick f_{k+1} in $I - (f_1, \dots, f_k)$ with minimal degree.

We have $\deg f_k \leq \deg f_{k+1}$ for all k : the case $k = 1$ was checked before, and for $k \geq 2$, f_k and f_{k+1} are in $I - (f_1, \dots, f_{k-1})$ so $\deg f_k \leq \deg f_{k+1}$ by the minimality condition on $\deg f_k$.

For $k \geq 1$, let $d_k = \deg f_k$ and c_k be the leading coefficient of f_k , so $d_k \leq d_{k+1}$ and $f_k(X) = c_k X^{d_k} + \text{lower-degree terms}$.

The ideal (c_1, c_2, \dots) in R (an ideal of leading coefficients) is finitely generated since R is Noetherian. Each element in this ideal is an R -linear combination of finitely many c_k , so $(c_1, c_2, \dots) = (c_1, \dots, c_m)$ for some m .

Since $c_{m+1} \in (c_1, c_2, \dots, c_m)$, we have

$$(3.2) \quad c_{m+1} = \sum_{k=1}^m r_k c_k$$

for some $r_k \in R$. From the inequalities $d_k \leq d_{m+1}$ for $k \leq m$, the leading term in $f_k(X) = c_k X^{d_k} + \dots$ can be moved into degree d_{m+1} by using $f_k(X)X^{d_{m+1}-d_k} = c_k X^{d_{m+1}} + \dots$, and this is in I since $f_k(X) \in I$ and I is an ideal in $R[X]$. By (3.2), the R -linear combination

$$\sum_{k=1}^m r_k f_k(X) X^{d_{m+1}-d_k}$$

is in the ideal (f_1, \dots, f_m) and its coefficient of $X^{d_{m+1}}$ is $\sum_{k=1}^m r_k c_k$, which equals the leading coefficient c_{m+1} of $f_{m+1}(X)$ in degree d_{m+1} . The difference

$$(3.3) \quad f_{m+1}(X) - \sum_{k=1}^m r_k f_k(X) X^{\deg f_{m+1} - \deg f_k}$$

is in I , it is *not* 0 since $f_{m+1} \in I - (f_1, \dots, f_m)$, and it has degree *less than* d_{m+1} since the terms $c_{m+1} X^{d_{m+1}}$ cancel out. But $f_{m+1}(X)$ has minimal degree among polynomials in $I - (f_1, \dots, f_m)$, and (3.3) is in $I - (f_1, \dots, f_m)$ with lower degree than d_{m+1} . That’s a contradiction. Thus I is finitely generated. \square

To summarize this proof in a single phrase, “use an ideal of leading coefficients”.

Remark 3.6. Our proof of the Hilbert Basis Theorem was by contradiction, so it is not constructive. A constructive proof runs as follows. For $R \neq 0$, I a nonzero ideal in $R[X]$, and $n \geq 0$, let L_n be the set of leading coefficients of polynomials in I of degree at most n together with 0. This is an ideal in R by the way polynomials add and get scaled by R . (While L_n might be (0) for small n , $L_n \neq (0)$ for large n since I contains a nonzero

polynomial and multiplying that by powers of X gives us polynomials in I of all higher degrees.) Since $L_n \subset L_{n+1}$, the ideals $\{L_n\}$ in R stabilize at some point, say $L_n = L_m$ for $n \geq m$. (Thus L_m is generated by the leading coefficients of *all* nonzero polynomials in I , so we could have defined L_m that way.) Each L_n has finitely many generators. When $L_n \neq (0)$, let P_n be a finite set of polynomials of degree at most n in I whose leading coefficients generate L_n . The union of the finite sets P_n for $n \leq m$ where $L_n \neq (0)$ is a generating set for I [3, Sect. 7.10]. This way of proving Hilbert's basis theorem essentially is due to Artin, according to van der Waerden [5].

Where in the proof of Theorem 3.5 did we use the assumption that R is Noetherian? It is how we know the ideals (c_1, \dots, c_k) for $k \geq 1$ stabilize for large k , so $c_{m+1} \in (c_1, \dots, c_m)$ for some m . The contradiction we obtain from that really shows $c_{m+1} \notin (c_1, \dots, c_m)$ for all m , so the proof of Theorem 3.5 could be viewed as proving the contrapositive: if $R[X]$ is not Noetherian then R is not Noetherian.

The converse of Theorem 3.5 is true: if the ring $R[X]$ is Noetherian then so is the ring R by Theorem 3.4, since $R \cong R[X]/(X)$.

Corollary 3.7. *If R is a Noetherian ring then so is $R[X_1, \dots, X_n]$.*

Proof. We induct on n . The case $n = 1$ is Theorem 3.5. For $n \geq 2$, write $R[X_1, \dots, X_n]$ as $R[X_1, \dots, X_{n-1}][X_n]$, with $R[X_1, \dots, X_{n-1}]$ being Noetherian by the inductive hypothesis, so we are reduced to the base case. \square

Remark 3.8. Corollary 3.7 when R is a field was proved by Hilbert in 1890 [2, Theorem 1, p. 474] as a pure existence theorem in a few pages, not by an algorithmic process.² This is what first made Hilbert famous in mathematics. Earlier, Gordan [1] settled the case $n = 2$ in 1868 by long calculations and spent 20 years unsuccessfully working on $n = 3$. Hilbert's proof for all n was revolutionary, as it illustrated the power of existence proofs over laborious constructive methods, and this became characteristic of much of modern mathematics. With the rise of fast computers in the late 20th century, generating sets for polynomial ideals can be computed routinely with Gröbner bases, which are a multivariable polynomial replacement for the Euclidean algorithm of polynomials in one variable.

Now we can build lots of Noetherian rings. The quadratic ring $\mathbf{Z}[\sqrt{d}]$ for a nonsquare integer d is Noetherian: it's isomorphic to $\mathbf{Z}[X]/(X^2 - d)$, $\mathbf{Z}[X]$ is Noetherian by Hilbert's basis theorem, and $\mathbf{Z}[X]/(X^2 - d)$ is Noetherian by Theorem 3.4. Similarly, $\mathbf{Z}[\sqrt{2}, \sqrt{3}]$ and $\mathbf{Z}[i, \sqrt[3]{2}, \sqrt[3]{10}]$ are Noetherian because they are isomorphic to $\mathbf{Z}[X, Y]/(X^2 - 2, Y^2 - 3)$ and $\mathbf{Z}[X, Y, Z]/(X^2 + 1, Y^3 - 2, Z^7 - 10)$. The ring $\mathbf{Z}[X, 1/X]$ is Noetherian since it is isomorphic to $\mathbf{Z}[X, Y]/(XY - 1)$.

For a field K and ideal I in $K[X_1, \dots, X_n]$, the ring $K[X_1, \dots, X_n]/I$ is Noetherian since K is trivially Noetherian. For instance, $\mathbf{R}[X, Y, Z]/(X^2 + Y^3 - Z^5, XYZ)$ is Noetherian.

Remark 3.9. In addition to polynomial rings in finitely many variables, formal power series rings in finitely many variables are important. For a Noetherian ring R the formal power series ring $R[[X_1, \dots, X_n]]$ is Noetherian, and as in the polynomial case writing $R[[X_1, \dots, X_n]]$ as $R[[X_1, \dots, X_{n-1}]][[X_n]]$ reduces the proof to the case $n = 1$. A formal power series usually doesn't have a leading coefficient, so the proof in the polynomial case doesn't work directly for power series. What can be used with formal power series instead of

²Hilbert could not use the proof that we gave for his basis theorem, since he didn't have the concept of a Noetherian ring in full generality available to him.

a leading term is a lowest degree term, so the proof of Theorem 3.4 can be adapted to formal power series by changing highest-degree coefficients into lowest-degree coefficients, although an infinite “limiting process” occurs in the proof since the multipliers on a generating set for the ideal will be power series. See [3, Theorem 7.11].

REFERENCES

- [1] P. Gordan, *Beweis, dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist*, J. Reine Angew. Mathematik **69** (1868), 323–354. Online at <https://eudml.org/doc/148066>.
- [2] D. Hilbert, *Ueber die Theorie der algebraischen Formen*, Math. Annalen **36** (1890), 473–534. Online at <https://eudml.org/doc/157506>.
- [3] N. Jacobson, “Basic Algebra II”, 2nd ed., W. H. Freeman & Co., New York, 1989.
- [4] E. Noether, *Idealtheorie in Ringbereichen*, Math. Annalen **83** (1921), 24–66. Online at <https://eudml.org/doc/158855>. English translation by D. Berlyne <https://arxiv.org/abs/1401.2577>.
- [5] B. L. van de Waerden, *On the sources of my book Moderne Algebra*, Historia. Math. **2** (1975), 31–40. Online at <https://core.ac.uk/download/pdf/82253306.pdf>.