REMARKS ABOUT EUCLIDEAN DOMAINS

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1. Introduction

The following definition of a Euclidean (not Euclidian!) domain is very common in textbooks. We write $\mathbf{N}$ for $\{0, 1, 2, \dots\}$.

Definition 1.1. An integral domain R is called *Euclidean* if there is a function $d\colon R-\{0\} \to \mathbf{N}$ with the following two properties:

(1) $d(a) \leq d(ab)$ for all nonzero a and b in R,
(2) for all a and b in R with $b \neq 0$ we can find q and r in R such that

$$a = bq + r, \quad r = 0 \text{ or } d(r) < d(b).$$

We will call (1) the *d-inequality*. Sometimes it is expressed in a different way: for nonzero x and y in R, if $x \mid y$ then $d(x) \leq d(y)$. This is equivalent to the d-inequality.

Examples of Euclidean domains are $\mathbf{Z}$ (with $d(n) = |n|$), $F[T]$ for a field F (with $d(f) = \deg f$; this example is *the* reason that one doesn't assume $d(0)$ is defined), and $\mathbf{Z}[i]$ ($d(\alpha) = \mathrm{N}(\alpha)$). A trivial kind of example is a field F with $d(a) = 1$ for all $a \neq 0$ (here all remainders are 0). For any Euclidean domain (R, d) in the sense of Definition 1.1, other examples are $(R, 2d)$, (R, d^2), and $(R, 2^d)$.

Definition 1.1 is used in [1, §2.1], [5, §37], [6, §7.2], [7, §6.5], [9, §3.7], [10, Chap. III §3], and [14, §18], which shows that Definition 1.1 accounts for all the basic examples (since otherwise we wouldn't see Definition 1.1 so widely used). However, we find a different definition in [4, §8.1], where the d-inequality is missing:

Definition 1.2. An integral domain R is called *Euclidean* if there is a function $d\colon R-\{0\} \to \mathbf{N}$ such that R has division with remainder with respect to d: for all a and b in R with $b \neq 0$ we can find q and r in R such that

$$a = bq + r, \quad r = 0 \text{ or } d(r) < d(b). \tag{1.1}$$

We allow $a = 0$ in this definition since in that case we can use $q = 0$ and $r = 0$.

A function $d\colon R - \{0\} \to \mathbf{N}$ that satisfies (1.1) will be called a *Euclidean function* on R. Thus a Euclidean domain in Definition 1.2 is an integral domain that admits a Euclidean function, while a Euclidean domain in Definition 1.1 is an integral domain that admits a Euclidean function satisfying the d-inequality.

Does Definition 1.2 describe a larger class of rings than Definition 1.1? No. We will show in Section 2 that every Euclidean domain (R, d) in the sense of Definition 1.2 can be equipped with a different Euclidean function $\widetilde{d}$ such that $\widetilde{d}(a) \leq \widetilde{d}(ab)$ for all a and b in R, so $(R, \widetilde{d})$ is Euclidean in the sense of Definition 1.1.

The main reason that the d-inequality is not included in the definition of a Euclidean domain in [4] is that it is irrelevant to prove the two main theorems about Euclidean

domains: that every Euclidean domain is a PID[1] and that the Euclidean algorithm in a Euclidean domain terminates after finitely many steps and produces a greatest common divisor. There can be greatest common divisors in rings that are not Euclidean (such as in $\mathbf{Z}[X, Y]$), but it may be hard in those rings to compute greatest common divisors by a method that avoids factorization. When a ring is Euclidean, the Euclidean algorithm in the ring lets us compute greatest common divisors without having to factor, which makes this method practical.

Why is the d-inequality nearly always mentioned in the (textbook) literature if it's actually not needed? Well, it is not needed for the two specific results cited in the previous paragraph, but it is convenient to use the d-inequality if we want to prove factorization into irreducibles in a Euclidean domain without proving the result more generally in a PID. There is factorization into irreducibles in every PID, which subsumes the same result for Euclidean domains since Euclidean domains are PIDs, but the proof of the existence of irreducible factorizations in a PID is less concrete than a proof available in Euclidean domains. We will see why in Sections 3 and 4.

In Section 5 we discuss Euclidean domains among quadratic rings.

2. Refining the Euclidean function

Suppose (R, d) is a Euclidean domain in the sense of Definition 1.2. We will introduce a new Euclidean function $\widetilde{d}\colon R - \{0\} \to \mathbf{N}$, built out of d, which satisfies $\widetilde{d}(a) \leq \widetilde{d}(ab)$. Then $(R, \widetilde{d})$ is Euclidean in the sense of Definition 1.1, so the rings that admit Euclidean functions in either sense are the same.

Here's the definition (trick?): for nonzero a in R, set

$$\widetilde{d}(a) = \min_{b \neq 0} d(ab).$$

That is, $\widetilde{d}(a)$ is the smallest d-value on the nonzero multiples of a. (We have $ab \neq 0$ when $b \neq 0$ since R is an integral domain.) Since $a = a \cdot 1$ is a nonzero multiple of a,

$$\widetilde{d}(a) \leq d(a) \tag{2.1}$$

for all nonzero a in R. For each $a \neq 0$ in R, $\widetilde{d}(a) = d(ab_0)$ for some nonzero b_0 and $d(ab_0) = \widetilde{d}(a) \leq d(ab)$ for all nonzero b. For example,

$$\widetilde{d}(1) = \min_{b \neq 0} d(b)$$

is the smallest d-value on $R - \{0\}$.

Theorem 2.1. *Let (R, d) be a Euclidean domain in the sense of Definition 1.2. Then $(R, \widetilde{d})$ is Euclidean in the sense of Definition 1.1.*

Proof. For nonzero a and b in R we have

$$\widetilde{d}(a) \leq \widetilde{d}(ab).$$

Indeed, write $\widetilde{d}(ab) = d(abc)$ for some nonzero c in R. Since abc is a nonzero multiple of a,

$$\widetilde{d}(a) \leq d(abc) = \widetilde{d}(ab).$$

[1]Not every PID is a Euclidean domain, as we'll see in Section 5.

We now show R admits division with remainder with respect to $\widetilde{d}$. Pick a and b in R with $b \neq 0$. Set $\widetilde{d}(b) = d(bc)$ for some nonzero $c \in R$. Using division of a by bc (which is nonzero) in (R, d) there are q_0 and r_0 in R such that

$$a = (bc)q_0 + r_0, \quad r_0 = 0 \text{ or } d(r_0) < d(bc).$$

Set $q = cq_0$ and $r = r_0$, so $a = bq + r$. If $r_0 = 0$ we are done, so we may assume $r_0 \neq 0$. Since $d(bc) = \widetilde{d}(b)$ and $\widetilde{d}(r) \leq d(r)$ (by (2.1)), the condition $d(r) = d(r_0) < d(bc)$ implies $\widetilde{d}(r) < \widetilde{d}(b)$. Thus

$$a = bq + r, \quad r = 0 \text{ or } \widetilde{d}(r) < \widetilde{d}(b).$$

Hence $(R, \widetilde{d})$ is a Euclidean domain in the sense of Definition 1.2. □

We end this section with a brief discussion of two other possible refinements one might want in a Euclidean function (but which we will not need later): uniqueness of the quotient and remainder it produces and multiplicativity.

In $\mathbf{Z}$ we write $a = bq + r$ with $0 \leq r < |b|$ and q and r are uniquely determined by a and b. There is also uniqueness of the quotient and remainder when we do division in $F[T]$ (relative to the degree function) and in a field (the remainder is always 0). Are there other Euclidean domains where the quotient and remainder are unique? Division in $\mathbf{Z}[i]$ does not have a unique quotient and remainder relative to the norm on $\mathbf{Z}[i]$. For instance, dividing $1 + 8i$ by $2 - 4i$ gives

$$1 + 8i = (2 - 4i)(-1 + i) - 1 + 2i \text{ and } 1 + 8i = (2 - 4i)(-2 + i) + 1 - 2i,$$

where both remainders have norm 5, which is less than $\mathrm{N}(2 - 4i) = 20$.

Theorem 2.2. *If R is a Euclidean domain where the quotient and remainder are unique then R is a field or $R = F[T]$ for a field F.*

Proof. See [11] or [12]. □

This might be a surprise: $\mathbf{Z}$ isn't in the theorem! Aren't the quotient and remainder unique there? Yes if we take the remainder r so that $0 \leq r < |b|$, but not if we try to fit it into the setting of Euclidean domains using $|r| < |b|$. For instance,

$$51 = 6 \cdot 8 + 3 \text{ and } 51 = 6 \cdot 9 - 3.$$

The point is that using the Euclidan function on the remainder too, in the case of $\mathbf{Z}$, permits negative remainders so in fact $\mathbf{Z}$ does not have a unique quotient and remainder when we measure the remainder by its absolute value.

The Euclidean function in many basic examples satisfies a stronger property than $d(a) \leq d(ab)$, namely $d(ab) = d(a)d(b)$ with $d(a) \geq 1$ when $a \neq 0$. For instance, this holds in $\mathbf{Z}$ ($d(n) = |n|$) and $\mathbf{Z}[i]$ ($d(\alpha) = \mathrm{N}(\alpha)$). The degree on $F[T]$ is not multiplicative, but $d(f) = 2^{\deg f}$ is multiplicative. There is also a multiplicative Euclidean function on a field F: $d(a) = 1$ for all $a \neq 0$. For an example of a Euclidean domain that does not admit a multiplicative Euclidean function, see [3].

3. Features of the d-inequality

Let R be a Euclidean domain. By Theorem 2.1 we may assume our Euclidean function d satisfies the d-inequality: $d(a) \leq d(ab)$ for all nonzero a and b in R. Using this inequality we will prove a few properties of d:

Theorem 3.1. *Let (R, d) be a Euclidean domain where d satisfies the d-inequality. Then*

(1) $d(a) \geq d(1)$ *for all nonzero* $a \in R$,
(2) *if* $b \in R^\times$ *then* $d(ab) = d(a)$ *for all nonzero* a,
(3) *if* $b \notin R^\times$ *and* $b \neq 0$ *then* $d(ab) > d(a)$ *for all nonzero* a.

In particular, for nonzero a *and* b, $d(ab) = d(a)$ *if and only if* $b \in R^\times$.

Proof. (1): By the d-inequality, $d(1) \leq d(1 \cdot a) = d(a)$.

(2): By the d-inequality, $d(a) \leq d(ab)$. To get the reverse inequality, let c be the inverse of b, so the d-inequality implies

$$d(ab) \leq d((ab)c) = d(a).$$

(3): We want to show the inequality $d(a) \leq d(ab)$ is strict when b is not a unit and not 0. The proof is by contradiction. Assume $d(a) = d(ab)$. Now use division of a by ab:

$$a = (ab)q + r, \quad r = 0 \text{ or } d(r) < d(ab).$$

We rewrite this as

$$a(1 - bq) = r, \quad r = 0 \text{ or } d(r) < d(a).$$

Since b is not a unit, $1 - bq$ is nonzero, so $a(1 - bq)$ is nonzero (because R is an integral domain). Thus $r \neq 0$, so the inequality $d(r) < d(a)$ becomes

$$d(a(1 - bq)) < d(a).$$

But this contradicts the d-inequality, which says $d(a) \leq d(a(1 - bq))$. Thus it is impossible for $d(a)$ to equal $d(ab)$ when b is not a unit. □

Note Theorem 3.1 is not saying two nonzero elements of R that have the same d-value are unit multiples, but rather that a multiple of a nonzero $a \in R$ has the same d-value as a if and only if it is a unit multiple of a. For example, in $\mathbf{Z}[i]$ we have $\mathrm{N}(1 + 2i) = \mathrm{N}(1 - 2i)$ but $1 + 2i$ and $1 - 2i$ are not unit multiples. Remember this!

Corollary 3.2. *Let* (R, d) *be a Euclidean domain where* d *satisfies the* d*-inequality. We have* $d(a) = d(1)$ *if and only if* $a \in R^\times$. *That is, the elements of least* d*-value in* R *are precisely the units.*

Proof. Take $a = 1$ in parts 2 and 3 of Theorem 3.1. □

We can see Corollary 3.2 working in $\mathbf{Z}$ and $F[T]$: the integers satisfying $|n| = |1|$ are ± 1, which are the units of $\mathbf{Z}$. The polynomials f in $F[T]$ satisfying $\deg f = \deg 1 = 0$ are the nonzero constants, which are the units of $F[T]$.

Corollary 3.3. *Let* (R, d) *be a Euclidean domain where* d *satisfies the* d*-inequality. If* a *and* b *are nonunits, then* $d(a)$ *and* $d(b)$ *are both less than* $d(ab)$.

Proof. This is immediate from part (3) of Theorem 3.1, where we switch the roles of a and b to get the inequality on both $d(a)$ and $d(b)$. □

4. Irreducible factorization

To see the simplicity introduced by a Euclidean function, we will prove irreducible factorization in both Euclidean domains and in PIDs.

Definition 4.1. Let R be an integral domain. A nonzero element a of R is called *irreducible* if it is not a unit and in every factorization $a = bc$, one of the factors b or c is a unit. A nonzero nonunit that is not irreducible is called *reducible*.

There are three types of nonzero elements in an integral domain: units (the invertible elements, whose factors are always units too), irreducibles (nonunits whose factorizations into two parts always involve one unit factor), and reducibles (nonunits that admit some factorization into a product of two nonunits). Notice that in a field there are no reducible or irreducible elements: everything is zero or a unit. So if we want to prove a theorem about irreducible factorization, we avoid fields.

Theorem 4.2. *In a Euclidean domain that is not a field, every nonzero nonunit is a product of irreducibles.*

Proof. Let (R, d) be a Euclidean domain that is not a field. By Theorem 2.1 we may assume $d(a) \leq d(ab)$ for all nonzero a and b in R. Therefore Corollary 3.3 applies.

We will prove the existence of irreducible factorizations by induction on the d-value. From Corollary 3.2, the units of R have the smallest d-value. An $a \in R$ with second smallest d-value must be irreducible: if we write $a = bc$ and b and c are both nonunits, then $d(b)$ and $d(c)$ are both less than $d(a)$ by Corollary 3.3. Therefore b and c are units, so a is a unit. This is a contradiction.

Assume now that $a \in R$ is a nonunit and all nonunits with smaller d-value admit an irreducible factorization. To prove a admits an irreducible factorization too, we may suppose a is not irreducible itself. Therefore there is some factorization $a = bc$ with b and c both nonunits. Then $d(b) < d(a)$ and $d(c) < d(a)$ by Corollary 3.3, so b and c both have irreducible factorizations by induction. Thus their product a has an irreducible factorization. □

The conclusion of Theorem 4.2 is true for PIDs, but the proof will require more abstract methods than induction.

Theorem 4.3. *In a* PID *that is not a field, every nonzero nonunit is a product of irreducibles.*

The proof of Theorem 4.3 relies on the following lemma, which has a recursive flavor.

Lemma 4.4. *If R is an integral domain and $a \in R$ is a nonzero nonunit that does not admit a factorization into irreducibles then there is a strict inclusion of principal ideals $(a) \subset (b)$ where b is some other nonzero nonunit that does not admit a factorization into irreducibles.*

Proof. By hypothesis a is not irreducible, so (since it is neither 0 nor a unit either) there is some factorization $a = bc$ where b and c are nonunits (and obviously are not 0 either). If both b and c admitted irreducible factorizations then so does a, so at least one of b or c has no irreducible factorization. Without loss of generality it is b that has no irreducible factorization. Since c is not a unit, the inclusion $(a) \subset (b)$ is strict. □

Now we can prove Theorem 4.3.

Proof. Suppose there is an element a in the PID that is not 0 or a unit and has no irreducible factorization. Then by Lemma 4.4 there is a strict inclusion

$$(a) \subset (a_1)$$

where a_1 has no irreducible factorization. Then using a_1 in the role of a (and Lemma 4.4 again) there is a strict inclusion

$$(a_1) \subset (a_2)$$

where a_2 has no irreducible factorization. This argument (repeatedly applying Lemma 4.4 to the generator of the next larger principal ideal) leads to an infinite increasing chain of principal ideals

$$(a) \subset (a_1) \subset (a_2) \subset (a_3) \subset \cdots \tag{4.1}$$

where all inclusions are strict. This turns out to be impossible in a PID.

Indeed, suppose a PID contains an infinite strictly increasing chain of ideals:

$$I_0 \subset I_1 \subset I_2 \subset I_3 \subset \cdots$$

and set

$$I = \bigcup_{n \geq 0} I_n.$$

This union I is an ideal. The reason is that the I_n's are strictly increasing, so every *finite* set of elements from I lies in a common I_n. (This is the key idea.) So I is closed under addition and arbitrary multiplications from the ring since each I_n has these properties. (Make sure you understand that step.) Because we are in a PID, I is principal: $I = (r)$ for some r in the ring. But because I is the union of the I_n's, r is in some I_N. Then $(r) \subset I_N$ since I_N is an ideal, so

$$I = (r) \subset I_N \subset I,$$

which means

$$I_N = I.$$

But this is impossible because the inclusion $I_{N+1} \subset I$ becomes $I_{N+1} \subset I_N$ and we were assuming I_N was a proper subset of I_{N+1}. Because of this contradiction, nonzero nonunits in a PID without an irreducible factorization do not exist. □

The proof that a PID does not contain an infinite strictly increasing chain of ideals holds for a broader class of rings than PIDs.

Theorem 4.5. *A commutative ring in which every ideal is finitely generated does not contain an infinite strictly increasing chain of ideals.*

Proof. The second half of the proof of Theorem 4.3 works in this more general context. All we have to do is show the logic works when I is finitely generated rather than principal. The point is that if $I = (x_1, \ldots, x_m)$ then the finitely many x_i's all lie in some common I_N (because the I_n's are an increasing chain). And now the contradiction is obtained just as before: $I_N = I$ but then $I_{N+1} \subset I_N$, contradiction. □

Corollary 4.6. *In an integral domain where every ideal is finitely generated, every nonzero nonunit has an irreducible factorization.*

Proof. If there were an element a that is not 0 or a unit and that did not admit an irreducible factorization then, as in the proof of Theorem 4.3, we could produce an infinite strictly increasing chain of (principal) ideals. But there are no infinite strictly increasing chains of ideals in the ring, by Theorem 4.5. □

Definition 4.7. A commutative ring where every ideal is finitely generated is called a *Noetherian* ring.

These rings are named after Emmy Noether, who was one of the pioneers of abstract algebra in the first half of the 20th century. Their importance, as a class of rings, stems from the stability of the Noetherian property under many basic constructions. If R is a Noetherian ring, so is every quotient ring R/I (which may not be an integral domain even if R is), every polynomial ring $R[X]$ (and thus $R[X_1, \ldots, X_n]$ by induction on n, viewing this as $R[X_1, \ldots, X_{n-1}][X_n]$), and every formal power series ring $R[[X]]$ (and thus $R[[X_1, \ldots, X_n]]$). The PID property behaves quite badly, *e.g.*, if R is a PID other than a field then $R[X]$ is not a PID. For instance, $R[X, Y] = R[Y][X]$ is never a PID for an integral domain R. But if R is Noetherian then $R[X, Y]$ is Noetherian. Briefly, the property "ideals are finitely generated" of Noetherian rings is more robust than the property "ideals are singly generated" of PIDs.

Using this terminology, Corollary 4.6 says in every *Noetherian integral domain* each element other than 0 or a unit has an irreducible factorization. It is worth comparing the proof of this general result (Corollary 4.6) to the special proof we gave in the case of Euclidean domains, where the proof of irreducible factorizations is tied up with features of the Euclidean function on the ring.

In the context of unique factorization domains, it is the uniqueness of the factorization that lies deeper than the existence. We are not discussing uniqueness here, which most definitely does *not* hold in most Noetherian integral domains. That is, the existence of irreducible factorizations (for all nonzero nonunits) is not a very strong constraint, to the extent that most integral domains you meet in day-to-day practice in mathematics are Noetherian so their elements automatically have some factorization into irreducible elements. But there usually is not going to be a unique factorization into irreducible elements.

5. Euclidean and non-Euclidean quadratic rings

The main importance of Euclidean domains for an algebra course is that they give us examples of PIDs. Three points are worth noting:

- Aside from computational issues (as in the Euclidean algorithm) it is more useful to know whether or not a ring is a PID than whether or not it is Euclidean.
- There are methods that let one show certain kinds of integral domains are PIDs without knowing whether or not they are Euclidean.
- There are PIDs that are not Euclidean.

The simplest setting where one can find PIDs that are not Euclidean are found among the quadratic rings.

Definition 5.1. A *quadratic ring* is a ring of the form $\mathbf{Z}[\gamma]$ where γ is a complex number that is a root of a monic irreducible quadratic polynomial $T^2 + aT + b \in \mathbf{Z}[T]$. We call $\mathbf{Z}[\gamma]$ *real* if γ is real and *imaginary* otherwise.

For instance, the Gaussian integers $\mathbf{Z}[i]$ are an imaginary quadratic ring (associated to the polynomial $T^2 + 1$). The ring $\mathbf{Z}[(1 + \sqrt{5})/2]$ is real quadratic, with $(1 + \sqrt{5})/2$ a root of $T^2 - T - 1$. Using the formula $\gamma = (-a \pm \sqrt{a^2 - 4b})/2$ it can be shown that if a is even then $\mathbf{Z}[\gamma] = \mathbf{Z}[\sqrt{d}]$ where $d = (a/2)^2 - b \in \mathbf{Z}$ and if a is odd then $\mathbf{Z}[\gamma] = \mathbf{Z}[(1 + \sqrt{d})/2]$ where $d = a^2 - 4b$. Note the real quadratic rings are subrings of $\mathbf{R}$ (with $a^2 - 4b > 0$) and the imaginary quadratic rings are not ($a^2 - 4b < 0$). Abstractly, $\mathbf{Z}[\gamma] \cong \mathbf{Z}[T]/(T^2 + aT + b)$. Since $T^2 + aT + b$ is irreducible in $\mathbf{Z}[T]$ by definition and $T^2 + aT + b = (T + a/2)^2 + (a^2 - 4b)/4$, $a^2 - 4b$ is not 0.

Since $\gamma^2 = -a\gamma - b \in \mathbf{Z} + \mathbf{Z}\gamma$, by induction every power of γ is in $\mathbf{Z} + \mathbf{Z}\gamma$, so

$$\mathbf{Z}[\gamma] = \mathbf{Z} + \mathbf{Z}\gamma.$$

We can't necessarily "complete" the square and write a quadratic ring in the form $\mathbf{Z}[\sqrt{m}]$ for some integer m. For instance, $\mathbf{Z}[(1+\sqrt{5})/2] \neq \mathbf{Z}[\sqrt{m}]$ for all integers m; compare that with $\mathbf{Z}[(1+\sqrt{5})/2] \supset \mathbf{Z}[\sqrt{5}]$ and $\mathbf{Q}[(1+\sqrt{5})/2] = \mathbf{Q}[\sqrt{5}]$.

When γ is one root of $T^2 + aT + b$, the other root is $\overline{\gamma} = -a - \gamma$, which is called the *conjugate* of γ. More generally, for $\alpha = x + y\gamma$ in $\mathbf{Z}[\gamma]$, where x and y are in $\mathbf{Z}$, the *conjugate* of α is

$$\overline{\alpha} := x + y\overline{\gamma} = x - ay - y\gamma.$$

In the special case that $a = 0$ and we write $b = -m$, so γ is a root of $T^2 - m$ (a square root of m), we have $\overline{\gamma} = -\gamma$ and $\overline{x + y\gamma} = x - y\gamma$. The *norm* of α is defined to be

$$\mathrm{N}(\alpha) = \alpha\overline{\alpha} = x^2 - axy + by^2.$$

This is an integer, and turns out to be zero only for $\alpha = 0$. Notice the coefficients in the formula $x^2 - axy + by^2$ do not coincide exactly with those of the polynomial $T^2 + aT + b$; the a occurs with the opposite sign (which only matters when $a \neq 0$). If $\alpha = c$ is in $\mathbf{Z}$ then $\mathrm{N}(c) = c^2$. In particular, $\mathrm{N}(\pm 1) = 1$.

A direct calculation shows the norm is multiplicative:

$$\mathrm{N}(\alpha\beta) = \mathrm{N}(\alpha)\,\mathrm{N}(\beta).$$

Example 5.2. If $\gamma = \sqrt{2}$ then $\mathrm{N}(x + y\sqrt{2}) = x^2 - 2y^2$, which takes both positive and negative values, *e.g.*, $\mathrm{N}(3 + 5\sqrt{2}) = -41$.

Example 5.3. If $\gamma = \sqrt{m}$ for $m \in \mathbf{Z}$ then $\mathrm{N}(x + y\gamma) = x^2 - my^2$. This has both positive and negative values if $m > 0$ and only nonnegative values if $m < 0$.

Example 5.4. If $\gamma = (1+\sqrt{5})/2$, a root of $T^2 - T - 1$, then $\mathrm{N}(x + y\gamma) = x^2 + xy - y^2$.

Several "small" quadratic rings are Euclidean with the absolute value of the norm as a Euclidean function. For instance, $\mathbf{Z}[i]$, $\mathbf{Z}[\sqrt{2}]$, and $\mathbf{Z}[\sqrt{-2}]$ are all Euclidean using $d(\alpha) = |\mathrm{N}(\alpha)|$. (For an imaginary quadratic ring like $\mathbf{Z}[i]$ we can drop the absolute value sign: the norm is already nonnegative.) This leads to two questions about a quadratic ring: is it Euclidean (with respect to some Euclidean function d on $\mathbf{Z}[\gamma]$) and is it norm-Euclidean (*i.e.*, Euclidean using the particular function $d(\alpha) = |\mathrm{N}(\alpha)|$)?

For a quadratic ring to be norm-Euclidean is more precise than it being Euclidean: maybe there could be a Euclidean function on the ring even if the absolute value of its norm is not a Euclidean function. The first example of this was found by Clark [2] in 1994: he showed $\mathbf{Z}[(1+\sqrt{69})/2]$ is Euclidean, and it was already known not to be norm-Euclidean. The second such example was found by Harper [8]: he showed $\mathbf{Z}[\sqrt{14}]$ is Euclidean in 2004. It was known to be a PID since the 19th century, and was known not to be norm-Euclidean since the early 20th century.

Theorem 5.5. *The ring $\mathbf{Z}[\sqrt{14}]$ is not norm-Euclidean.*

Since Harper showed $\mathbf{Z}[\sqrt{14}]$ is Euclidean, the significance of Theorem 5.5 is that $\mathbf{Z}[\sqrt{14}]$ is not Euclidean for a specific function – $d(\alpha) = |\mathrm{N}(\alpha)|$ – even though it turns out to be Euclidean for some other function, which is described in the answer by Franz Lemmermeyer at https://math.stackexchange.com/questions/1148364.

Proof. We will show a specific equation can't be solved in $\mathbf{Z}[\sqrt{14}]$: $1+\sqrt{14} = 2\gamma + \rho$ where $|\,\mathrm{N}(\rho)| < |\,\mathrm{N}(2)| = 4$. Assume this has a solution $\gamma = m + n\sqrt{14}$ and $\rho = a + b\sqrt{14}$ in $\mathbf{Z}[\sqrt{14}]$, so

$$1 + \sqrt{14} = (2m + a) + (2n + b)\sqrt{14} \Longrightarrow a = 1 - 2m,\ b = 1 - 2n,$$

which means we'd want $|(1-2m)^2 - 14(1-2n)^2| < 4$, so

$$(2m-1)^2 - 14(2n-1)^2 = 0, \pm 1, \pm 2, \pm 3.$$

The left side is odd, so the right is ± 1 or ± 3. Since odd numbers square to 1 mod 8, $(2m-1)^2 - 14(2n-1)^2 \equiv 1 - 14 \equiv 3 \bmod 8$, so

$$(2m-1)^2 - 14(2n-1)^2 = 3.$$

Reducing both sides modulo 7, we get $3 \equiv \square \bmod 7$. However, the nonzero squares modulo 7 are 1, 2, and 4. Thus we have a contradiction. □

There are only finitely many *norm-Euclidean* quadratic rings[2] but we expect there are infinitely many Euclidean quadratic rings: most quadratic rings that are Euclidean should not be norm-Euclidean.

The rest of this handout is concerned with setting up the background to prove a particular imaginary quadratic ring is a PID but is not Euclidean (Theorem 5.13).

Theorem 5.6. *In a quadratic ring $\mathbf{Z}[\gamma]$, the units are the elements with norm ± 1.*

Proof. If $\alpha\beta = 1$ in $\mathbf{Z}[\gamma]$ then taking norms of both sides shows $\mathrm{N}(\alpha)\,\mathrm{N}(\beta) = \mathrm{N}(1) = 1$ in $\mathbf{Z}$, so $\mathrm{N}(\alpha) = \pm 1$. Conversely, if $\mathrm{N}(\alpha) = \pm 1$ then $\alpha\overline{\alpha} = \pm 1$, so α is invertible (with inverse $\pm\overline{\alpha}$). □

Example 5.7. The units of $\mathbf{Z}[\sqrt{2}]$ are built from integral solutions to $x^2 - 2y^2 = \pm 1$. For instance, one solution is $x = 1$ and $y = 1$, giving the unit $1 + \sqrt{2}$. Its powers are also units (units are closed under multiplication), so $\mathbf{Z}[\sqrt{2}]$ has infinitely many units.

Example 5.8. Units in $\mathbf{Z}[\sqrt{3}]$ come from integral solutions to $x^2 - 3y^2 = \pm 1$. However, there are no solutions to $x^2 - 3y^2 = -1$ since the equation has no solutions modulo 3: $x^2 \equiv -1 \bmod 3$ has no solution. Thus the units of $\mathbf{Z}[\sqrt{3}]$ only correspond to solutions to $x^2 - 3y^2 = 1$. One nontrivial solution (that is, other than ± 1) is $x = 2$ and $y = 1$, which yields the unit $2 + \sqrt{3}$. Its powers give infinitely many more units.

Example 5.9. The units of $\mathbf{Z}[\sqrt{-2}]$ come from integral solutions to $x^2 + 2y^2 = 1$. The right side is at least 2 once $y \neq 0$, so the only integral solutions are $x = \pm 1$ and $y = 0$, corresponding to the units ± 1. In contrast to the previous two examples, where there are infinitely many units, $\mathbf{Z}[\sqrt{-2}]$ has only two units.

The following theorem about Euclidean domains is the key to proving certain (imaginary) quadratic rings are not Euclidean. Notice the proof does not require the Euclidean function on the ring to satisfy the d-inequality.

Theorem 5.10. *Let (R, d) be a Euclidean domain that is not a field, so there is a nonunit $a \in R$ with least d-value among all non-units. Then the quotient ring $R/(a)$ is represented by 0 and units.*

[2]See https://math.stackexchange.com/questions/2710457/norm-euclidean-fields.

Proof. Pick $x \in R$. By division with remainder in R we can write $x = aq + r$ where $r = 0$ or $d(r) < d(a)$. If $r \neq 0$, then the inequality $d(r) < d(a)$ forces r to be a unit. Since $x \equiv r \bmod a$, we conclude that $R/(a)$ is represented by 0 and by units. □

Example 5.11. When $R = \mathbf{Z}$ we can use $a = 2$. Then $\mathbf{Z}/2\mathbf{Z}$ is represented by 0 and 1. When $R = \mathbf{Z}[i]$ we can use $a = 1 + i$. Then $\mathbf{Z}[i]/(1 + i)$ is represented by 0 and 1 as well as by 0 and i. These examples show some units could be congruent modulo a, but at least every element of the ring is congruent modulo a to 0 or some (perhaps more than one) unit.

We have shown that if R is a Euclidean domain that is not a field, then there are elements of R (namely nonunits with least d-value) modulo which everything is congruent to 0 or a unit from R. A domain that's not a field and which has no element modulo which everything is congruent to 0 or a unit from R therefore can't be a Euclidean domain.

Remark 5.12. In a domain R, an element a for which the ring $R/(a)$ is represented by 0 and units in R is called a *universal side divisor* in the literature. This terminology seems strange. What's a side divisor? Remember the property, but forget the label (and don't use it, because nobody will know what you're talking about).

Theorem 5.13. *The quadratic ring* $R = \mathbf{Z}[(1 + \sqrt{-19})/2]$ *is a* PID *and not Euclidean.*

Proof. We will only sketch the proof that R is a PID. Here are two methods.

- It can be shown for nonzero x and y in R that either $y \mid x$ or $0 < \mathrm{N}(ax+by) < \mathrm{N}(y)$, where N is the norm map on R. This property of R implies it is a PID by a slight modification of the proof that a Euclidean domain is a PID. For further details, see [4, p. 282] or [15, Sect. 2].
- From methods of algebraic number theory (the "Minkowski bound"), if there is a non-principal ideal in R then there is such an ideal in R with index less than $2\sqrt{19}/\pi \approx 2.7$, so the index has to be 2 (the index can't be 1, since the ideal is then $R = (1)$, which is principal). It can be shown R has no ideal with index 2, principal or non-principal, by using that $(1 + \sqrt{-19})/2$ is a root of $X^2 - X + 5$ and $X^2 - X + 5 \bmod 2$ has no roots in $\mathbf{Z}/(2)$. The details justifying this can be found in an algebraic number theory text in the part where class numbers are computed (R has class number one).

To prove R is not Euclidean (this is stronger than R not being norm-Euclidean!) we first note R is not a field since $\mathbf{Z} \subset R$ but $1/2 \notin R$ (in fact, $R \cap \mathbf{Q} = \mathbf{Z}$). Therefore to prove R is not Euclidean we will show R doesn't satisfy the conclusion of Theorem 5.10: for no nonunit $a \in R$ is $R/(a)$ represented by 0 and units.

First we compute the norm of a typical element $\alpha = x + y(1 + \sqrt{-19})/2$:

$$\mathrm{N}(\alpha) = x^2 + xy + 5y^2 = \left(x + \frac{y}{2}\right)^2 + \frac{19y^2}{4}. \tag{5.1}$$

This norm always takes values ≥ 0 (this is clearer from the second expression for it than the first) and once $y \neq 0$ we have $\mathrm{N}(\alpha) \geq 19y^2/4 \geq 19/4 > 4$. In particular, the units are solutions to $\mathrm{N}(\alpha) = 1$, which are ± 1:

$$R^\times = \{\pm 1\}.$$

The first few norm values are 0, 1, 4, 5, 7, and 9. In particular, there is no element of R with norm 2 or 3. This and the fact that $R^\times \cup \{0\}$ has size 3 are the key facts we will use.

If R were Euclidean then there would be a nonunit a in R such that $R/(a)$ is represented by 0 and units, so by 0, 1, or -1. Perhaps $1 \equiv -1 \bmod a$, but we definitely have $\pm 1 \not\equiv 0 \bmod a$ (since a is not a unit, (a) is a proper ideal so $R/(a)$ is not the zero ring). Thus $R/(a)$ has size 2 (if $1 \equiv -1 \bmod a$) or 3 (if $1 \not\equiv -1 \bmod a$). We show this can't happen.

If $R/(a)$ has size 2 then $2 \equiv 0 \bmod a$ (think about $R/(a)$ as an additive group of size 2), so $a \mid 2$ in R. Therefore $\mathrm{N}(a) \mid 4$ in $\mathbf{Z}$. There are no elements of R with norm 2, so the only nonunits with norm dividing 4 are elements with norm 4. A check using (5.1) shows the only such numbers are ± 2. However, $R/(2) = R/(-2)$ does not have size 2: it has size 4, since $R \cong \mathbf{Z}^2$ as an abelian group implies $R/2R \cong (\mathbf{Z}/2\mathbf{Z})^2$.

Similarly, if $R/(a)$ has size 3 then $a \mid 3$ in R, so $\mathrm{N}(a) \mid 9$ in $\mathbf{Z}$. There is no element of R with norm 3, so a must have norm 9 (it doesn't have norm 1 since it is not a unit). The only elements of R with norm 9 are ± 3, so $a = \pm 3$. The ring $R/(3) = R/(-3)$ does not have size 3: its size is 9 since $R/3R \cong (\mathbf{Z}/3\mathbf{Z})^2$.

Since $R^\times \cup \{0\}$ has size 3 and R has no element a such that $R/(a)$ has size 2 or 3, R can't be a Euclidean domain. □

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