

WELL-DEFINED FUNCTIONS

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1. INTRODUCTION

When defining an operation or function or formula, care has to be taken if the expression for the operation or function or formula involves making choices along the way.

Example 1.1. In elementary school we learned how to add fractions by a rule amounting to

$$(1.1) \quad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

For instance,

$$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}, \quad \frac{2}{4} + \frac{1}{3} = \frac{10}{12} = \frac{5}{6}.$$

What is not emphasized in elementary school is that changing the numerator and denominator without changing the fraction (e.g., writing $1/2$ as $2/4$) does *not* change the answer. All students implicitly accept this, but for a rigorous development of fractions this feature should be explained: if $a/b = a'/b'$ and $c/d = c'/d'$ why is $(ad + bc)/bd = (a'd' + b'c')/(b'd')$?

Here is how this shown. From $a/b = a'/b'$ and $c/d = c'/d'$ we have $ab' = a'b$ and $cd' = c'd$ in \mathbf{Z} . To show $(ad + bc)/bd = (a'd' + b'c')/(b'd')$ we check that $(ad + bc)(b'd') = (a'd' + b'c')(bd)$:

$$(ad + bc)(b'd') = (ab')dd' + bb'(cd') = (a'b)dd' + bb'(c'd) = (a'd' + b'c')(bd). \quad \checkmark$$

Those who have poor algebra skills may try to add fractions using the rule

$$(1.2) \quad \frac{a}{b} \oplus \frac{c}{d} = \frac{a + c}{b + d},$$

where we write \oplus rather than $+$ to emphasize that this is a different operation. This operation doesn't make sense (ignore that it isn't addition of fractions), since its output can change if we change a numerator and denominator without changing the fraction, e.g.,

$$\frac{1}{2} \oplus \frac{1}{3} = \frac{2}{5}, \quad \frac{2}{4} \oplus \frac{1}{3} = \frac{3}{7},$$

and $2/5 \neq 3/7$.

We say that $+$ in (1.1) is a *well-defined* operation on fractions while (1.2) is not a well-defined operation on fractions. If you change the way you write fractions (different numerators and denominators, but keeping the same values of the fractions), the answer does not change in (1.1) but it does change in (1.2).¹

¹We could make (1.2) well-defined by insisting the fractions a/b and c/d are in reduced form with positive denominators. A fraction can be written like that in just one way. The resulting operation on fractions, at least those that are positive, is called the median and occurs in the study of Farey fractions and continued fractions.

Example 1.2. In calculus, $\int_a^b f(x)dx$ can be computed as

$$\int_a^b f(x)dx = F(b) - F(a),$$

where $F(x)$ is an arbitrary anti-derivative of $f(x)$ on $[a, b]$, *i.e.*, $F'(x) = f(x)$ for all x in $[a, b]$. This formula for $\int_a^b f(x)dx$ involves a choice of anti-derivative for $f(x)$, but the formula does *not* depend on the choice: every anti-derivative $G(x)$ of $f(x)$ differs from $F(x)$ by a constant, say $G(x) = F(x) + C$, and using this other anti-derivative does not change the difference:

$$G(b) - G(a) = (F(b) + C) - (F(a) + C) = F(b) - F(a).$$

That is, differences of the values of an anti-derivative of $f(x)$ at $x = a$ and $x = b$ are independent of the choice of anti-derivative of $f(x)$ on the interval $[a, b]$.²

In contrast, the “rule” $F(b) + F(a)$ depends on the choice of anti-derivative of $f(x)$, since

$$G(b) + G(a) = (F(b) + C) + (F(a) + C) = F(b) + F(a) + 2C,$$

which is a new value if $C \neq 0$. Taking differences in an anti-derivative cancels the effect of the undetermined additive constant, so the expression $F(b) - F(a)$ is a well-defined value based on the original input $f(x)$ and the interval $[a, b]$.

2. WELL-DEFINEDNESS IN MODULAR ARITHMETIC

In algebra, there are many times when we want to define a function on $\mathbf{Z}/(m)$, the integers modulo m . Usually the value of the function on a congruence class mod m involves a *choice* of representative for that congruence class, and we must check this is well-defined, *i.e.*, the function’s value does not depend on the choice of representative.

Example 2.1. If $a \equiv b \pmod{2}$, then $(-1)^a = (-1)^b$. Therefore, we have a *well-defined* function $f: \mathbf{Z}/(2) \rightarrow \{\pm 1\}$ given by the rule

$$f(a \pmod{2}) = (-1)^a.$$

You need to train yourself to check that functions you want to define on $\mathbf{Z}/(m)$ in terms of congruence class representatives mod m are in fact well-defined (that is, independent of the choices made). The following table lists examples of well-defined functions on various $\mathbf{Z}/(m)$ ’s. Some functions that are not well-defined, *i.e.*, are nonsense, will be given later.

Group	Rule
$\mathbf{Z}/(2)$	$(-1)^a$
$\mathbf{Z}/(4)$	$(-1)^a$
$\mathbf{Z}/(4)$	i^a
$\mathbf{Z}/(12)$	i^a
$\mathbf{Z}/(4)$	$2^a \pmod{5}$
$\mathbf{Z}/(3)$	$2^a \pmod{7}$
$\mathbf{Z}/(3)$	$a^3 \pmod{9}$
$\mathbf{Z}/(4)$	$\frac{a(a-1)}{2} \pmod{2}$

²This is why in physics, potential energy has no intrinsic meaning (the zero level of potential energy can be anywhere), but *differences* in potential energy are physically meaningful.

In all cases, to show a function $f: \mathbf{Z}/(m) \rightarrow S$ that takes values in some set S is well-defined (that is, makes sense), we must show

$$a \equiv b \pmod{m} \implies f(a \pmod{m}) = f(b \pmod{m}).$$

Let's run through this kind of check for the examples in the table above.

- if $a \equiv b \pmod{2}$, then $a = b + 2k$ for $k \in \mathbf{Z}$, so $(-1)^a = (-1)^{b+2k} = (-1)^b(-1)^{2k} = (-1)^b$.
- if $a \equiv b \pmod{4}$, then $a = b + 4k$ for $k \in \mathbf{Z}$, so $(-1)^a = (-1)^{b+4k} = (-1)^b(-1)^{4k} = (-1)^b$.
- if $a \equiv b \pmod{4}$, then $a = b + 4k$ for $k \in \mathbf{Z}$, so $i^a = i^{b+4k} = i^b i^{4k} = i^b (i^4)^k = i^b$.
- if $a \equiv b \pmod{12}$, then $a = b + 12k$ for $k \in \mathbf{Z}$, so $i^a = i^{b+12k} = i^b i^{12k} = i^b (i^4)^{3k} = i^b$.
- if $a \equiv b \pmod{4}$, then $a = b + 4k$ for $k \in \mathbf{Z}$, so $2^a = 2^{b+4k} = 2^b (2^4)^k \equiv 2^b \pmod{5}$ since $2^4 = 16 \equiv 1 \pmod{5}$.
- if $a \equiv b \pmod{3}$, then $a = b + 3k$ for $k \in \mathbf{Z}$, so $2^a = 2^{b+3k} = 2^b (2^3)^k \equiv 2^b \pmod{7}$ since $2^3 = 8 \equiv 1 \pmod{7}$.
- if $a \equiv b \pmod{3}$, then $a = b + 3k$ for $k \in \mathbf{Z}$, so $a^3 = (b + 3k)^3 = b^3 + 3b^2(3k) + 3b(3k)^2 + (3k)^3 = b^3 + 9(b^2k + 3bk^2 + 3k^3) \equiv b \pmod{9}$.
- if $a \equiv b \pmod{4}$, then $a(a-1) \equiv b(b-1) \pmod{4}$. Both $a(a-1)$ and $b(b-1)$ are even, since a or $a-1$ is even and b or $b-1$ is even. Therefore we can divide both sides of the congruence and the modulus by 2 to obtain $\frac{a(a-1)}{2} \equiv \frac{b(b-1)}{2} \pmod{2}$.

Before looking at more complicated examples that show how to check a function on $\mathbf{Z}/(m)$ is well-defined, let's see why some "functions" on $\mathbf{Z}/(m)$ are *not* well-defined.

Group	Bad Rule
$\mathbf{Z}/(3)$	$(-1)^a$
$\mathbf{Z}/(5)$	$2^a \pmod{5}$
$\mathbf{Z}/(3)$	$a^2 \pmod{9}$
$\mathbf{Z}/(2)$	$\frac{a(a-1)}{2} \pmod{2}$

To show a rule introduced on $\mathbf{Z}/(m)$ is not well-defined, we just need to find an example of a and b with $a \equiv b \pmod{m}$ and the rule leads to different values in the two cases:

- $(-1)^a$ is not a well-defined function on $\mathbf{Z}/(3)$, since $4 \equiv 1 \pmod{3}$, but $(-1)^4 = 1$ while $(-1)^1 = -1$.
- $2^a \pmod{5}$ is not a well-defined function on $\mathbf{Z}/(5)$ since $5 \equiv 0 \pmod{5}$ and $2^5 = 32 \equiv 2 \pmod{5}$ while $2^0 \equiv 1 \pmod{5}$.
- $5 \equiv 2 \pmod{3}$, but $5^2 = 25 \equiv 7 \pmod{9}$ and $2^2 \equiv 4 \pmod{9}$. (Compare this with cubing, as in the first table: $5^3 \equiv 2^3 \pmod{9}$.)
- $5 \equiv 3 \pmod{2}$ and $\frac{5(5-1)}{2} \equiv 0 \pmod{2}$ while $\frac{3(3-1)}{2} \equiv 1 \pmod{2}$. (Compare with the first table, where we saw $\frac{a(a-1)}{2} \pmod{2}$ is well-defined when we treat a as belonging to $\mathbf{Z}/(4)$ instead of $\mathbf{Z}/(2)$.)

While showing a function is not well-defined requires just one counterexample, as above, to show a function *is* well-defined requires more work than checking one example! You must verify that the rule for the function is truly independent of the representative you pick from the congruence class, for all congruence classes. Look again at the proofs above that the various well-defined functions given earlier in the first table are in fact well-defined.

There is a point of terminology worth noting here. Speaking about "a well-defined function" is in some sense an *abuse of language*, since a function has to make sense before it

is really called a function in the first place. However, the conventional manner of writing and speaking is to introduce a formula on some $\mathbf{Z}/(m)$ and call it a function before right away doing the check that it is well-defined (that is, it is independent of the choice of representative) and thus worthy of being called a function.

Now that we have seen how to check a function is or is not well-defined for some explicit choices of the modulus, we consider some situations where the modulus is a parameter (it is not fixed to a specific value).

Theorem 2.2. *The function $\mathbf{Z}/(m) \rightarrow \mathbf{Z}$ given by $f(a \bmod m) = (a, m)$ is well-defined. In other words, the greatest common divisor of an integer with m does not change if we adjust the integer modulo m .*

Let's see what the theorem is saying in an example before we prove the theorem. In $\mathbf{Z}/(10)$ the congruence class of 4 is a list of integers $\{\dots, -6, 4, 14, 24, \dots\}$ and every single integer in this congruence class has the same gcd with 10: $\gcd(4, 10) = 2$, $\gcd(14, 10) = 2$, $\gcd(-6, 10) = 2$, and so on. If $a \equiv 4 \pmod{10}$ then $\gcd(a, 10) = 2$.

Proof. Let $a \equiv b \pmod{m}$, say $a = b + mk$ for $k \in \mathbf{Z}$. Set $d = (a, m)$ and $d' = (b, m)$. We will show $d = d'$.

Since d divides both a and m , it divides $a - mk = b$. Thus d is a common divisor of b and m , so $d \leq d'$.

Since d' divides both b and m , it divides $b + mk = a$. Thus d' is a common divisor of a and m , so $d' \leq d$.

Comparing the two inequalities on d and d' , we have $d = d'$. □

Theorem 2.3. *If G is a finite group and $g \in G$ satisfies $g^n = e$ for some $n \in \mathbf{Z}^+$ then the function $f: \mathbf{Z}/(n) \rightarrow G$ given by $f(a \bmod n) = g^a$ is well-defined.*

Proof. If $a \equiv b \pmod{n}$, write $a = b + nk$ for some $k \in \mathbf{Z}$. Then $g^a = g^{b+nk} = g^b(g^n)^k = g^b$ since $g^n = e$. □

The point of Theorem 2.3 is that, while we are used to viewing g^a with a as an integer, we can also view g^a as having the exponent a belong to $\mathbf{Z}/(n)$ if $g^n = e$.

Note we do not need n to be the order of g in Theorem 2.3. All that was used in the proof was the condition $g^n = e$, and that n might well be a multiple of the order. (Recall the example of $(-1)^a$ with $a \in \mathbf{Z}/(4)$.)

Theorem 2.4. *If $d \mid n$, the reduction map $r: \mathbf{Z}/(n) \rightarrow \mathbf{Z}/(d)$ given by the rule $r(a \bmod n) = a \bmod d$ is well-defined.*

Proof. If $a \equiv b \pmod{n}$, then $n \mid (a - b)$. Since $d \mid n$, we have $d \mid (a - b)$, so $a \equiv b \pmod{d}$. □

For example the function $r: \mathbf{Z}/(10) \rightarrow \mathbf{Z}/(5)$ where $r(a \bmod 10) = a \bmod 5$ is well-defined: if two integers are congruent mod 10 then they are also congruent mod 5.

The "function" $r: \mathbf{Z}/(10) \rightarrow \mathbf{Z}/(4)$ given by $r(a \bmod 10) = a \bmod 4$ is *not* well-defined (it is not a function at all), since $10 \equiv 0 \pmod{10}$ but $10 \not\equiv 0 \pmod{4}$.

3. MORE EXAMPLES OF WELL-DEFINEDNESS

In this last section we describe more instances where a definition is well-defined: rational powers, a projection formula, and arc length integrals.

Our first example is the definition of rational powers. Fix $a > 0$. For each $k \in \mathbf{Z}^+$, a has a unique positive k th root. This number is the unique positive solution to the equation

$x^k = a$ and it is often denoted as $\sqrt[k]{a}$. We define $a^{1/k} = \sqrt[k]{a}$, and for a fraction m/n where $m \in \mathbf{Z}$ and $n \in \mathbf{Z}^+$, we define $a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$. There is absolutely nothing ambiguous about the notation $a^{1/n}$, but there is something potentially ambiguous about writing $(\sqrt[n]{a})^m$ as $a^{m/n}$: a fraction can be written in many ways as a ratio of integers m/n (with $n > 0$), so we have to check that any two ways of writing a fraction as m/n leads to the same meaning for $(\sqrt[n]{a})^m$.

Theorem 3.1. *Fix a positive number a . If $m/n = m'/n'$ where n and n' are in \mathbf{Z}^+ and m and m' are in \mathbf{Z} , then $(\sqrt[n]{a})^m = (\sqrt[n']{a})^{m'}$.*

Proof. Set $u = (\sqrt[n]{a})^m$ and $v = (\sqrt[n']{a})^{m'}$. We want to show $u = v$. Since $m/n = m'/n'$ and the denominators are positive, the numerators have the same sign (positive or negative) or the numerators are both 0.

Case 1: $m = m' = 0$. We have $u = (\sqrt[n]{a})^0 = 1$ and $v = (\sqrt[n']{a})^0 = 1$.

Case 2: $m > 0$ and $m' > 0$. We will show $u^k = v^k$ for a certain positive integer k , and then it would follow that $u = v$ by the uniqueness of the k th root of positive numbers.

Since u involves an n th root and v involves an n' th root, we compute

$$u^n = ((\sqrt[n]{a})^m)^n = ((\sqrt[n]{a})^n)^m = a^m$$

and

$$v^{n'} = ((\sqrt[n']{a})^{m'})^{n'} = ((\sqrt[n']{a})^{n'})^{m'} = a^{m'}.$$

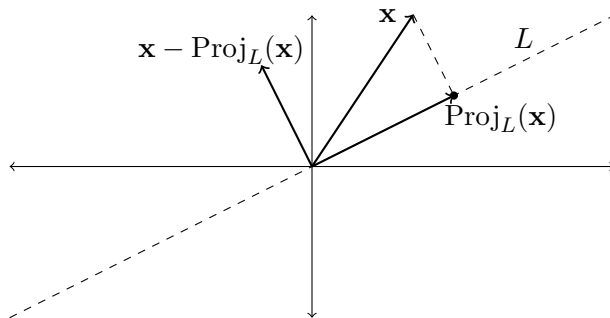
Since $m/n = m'/n'$, we have $nm' = n'm$, so we further compute

$$(u^n)^{m'} = (a^m)^{m'} = a^{mm'}, \quad (v^{n'})^m = (a^{m'})^m = a^{m'm} = a^{mm'}.$$

Both expressions are $a^{mm'}$, so $u^{nm'} = v^{n'm}$. The exponents here are equal in \mathbf{Z}^+ , so by the uniqueness of the k th root when $k = nm' = n'm$, we have $u = v$.

Case 3: $m < 0$ and $m' < 0$. From $m/n = m'/n'$ we have $(-m)/n = (-m')/n'$, so by Case 2, $(\sqrt[n]{a})^{-m} = (\sqrt[n']{a})^{-m'}$. Inverting both sides, $(\sqrt[n]{a})^m = (\sqrt[n']{a})^{m'}$. \square

Our next example is a projection formula in linear algebra. For a one-dimensional subspace L in \mathbf{R}^n and each $\mathbf{x} \in \mathbf{R}^n$, $\text{Proj}_L(\mathbf{x})$ denotes the vector in L for which the difference $\mathbf{x} - \text{Proj}_L(\mathbf{x})$ is perpendicular to L , as illustrated in the picture below ($n = 2$).



Calculations with dot products lead to a formula for this projection:

$$\text{Proj}_L(\mathbf{x}) = \left(\frac{\mathbf{x} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v}$$

where \mathbf{v} is a choice of nonzero vector in L . Let's check this formula does not depend on \mathbf{v} .

Theorem 3.2. *If L is a one-dimensional subspace of \mathbf{R}^n and \mathbf{v} is a nonzero vector lying in L then for each \mathbf{x} in \mathbf{R}^n , the vector*

$$\left(\frac{\mathbf{x} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v}$$

is independent of the choice of \mathbf{v} .

Proof. Let \mathbf{v} and \mathbf{w} be two nonzero vectors in L . Since L is one-dimensional, \mathbf{v} and \mathbf{w} are scalar multiples of each other, say $\mathbf{w} = c\mathbf{v}$ with $c \in \mathbf{R}$. Then $c \neq 0$, so

$$\left(\frac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \right) \mathbf{w} = \left(\frac{\mathbf{x} \cdot (c\mathbf{v})}{(c\mathbf{v}) \cdot (c\mathbf{v})} \right) (c\mathbf{v}) = \frac{c(\mathbf{x} \cdot \mathbf{v})}{c^2(\mathbf{v} \cdot \mathbf{v})} (c\mathbf{v}) = \frac{c^2(\mathbf{x} \cdot \mathbf{v})}{c^2(\mathbf{v} \cdot \mathbf{v})} \mathbf{v} = \left(\frac{\mathbf{x} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v}.$$

□

Our last example is arc length integrals. Suppose C is a “nice” parametrized curve in \mathbf{R}^n : there is a function $\gamma: [a, b] \rightarrow \mathbf{R}^n$ such that

- $C = \gamma([a, b])$,
- γ is continuously differentiable,
- $\gamma'(t) \neq \mathbf{0}$ for all t .

Using γ we define the length of C to be

$$\int_a^b \|\gamma'(t)\| dt.$$

We will show this notion of length is unaffected by a “change of variables”.

Theorem 3.3. *If $\varphi: [c, d] \rightarrow [a, b]$ is a continuously differentiable bijection with $\varphi' \neq 0$ then the composition $\tilde{\gamma} := \gamma \circ \varphi: [c, d] \rightarrow \mathbf{R}^n$ is also a “nice” parametrization of C and the length of C using $\tilde{\gamma}$ is the same³ as the length of C using γ :*

$$\int_a^b \|\gamma'(t)\| dt = \int_c^d \|\tilde{\gamma}'(u)\| du.$$

Proof. $\tilde{\gamma}$ is “nice” parametrization of C . By $C = \gamma([a, b])$ and $[a, b] = \varphi([c, d])$, we have $C = \gamma(\varphi([c, d])) = \tilde{\gamma}([c, d])$. A composition of continuously differentiable functions is continuously differentiable, so $\tilde{\gamma}$ is continuously differentiable. Using the chain rule on $\tilde{\gamma} = \gamma \circ \varphi$, we have $\tilde{\gamma}'(u) = (\gamma \circ \varphi)'(u) = \gamma'(\varphi(u))\varphi'(u)$, so $\tilde{\gamma}'(u) \neq \mathbf{0}$ since $\gamma'(\varphi(u)) \neq \mathbf{0}$ and the scalar $\varphi'(u)$ is nonzero.

Length is not changed. Since φ' is continuous and nonvanishing, either $\varphi' > 0$ on $[c, d]$ or $\varphi' < 0$ on $[c, d]$. If $\varphi' > 0$ then $a = \varphi(c)$, $b = \varphi(d)$, and $\|\tilde{\gamma}'(u)\| = \|\gamma'(\varphi(u))\|\varphi'(u) = \|\gamma'(\varphi(u))\|\varphi'(u)$, so

$$\int_c^d \|\tilde{\gamma}'(u)\| du = \int_c^d \|\gamma'(\varphi(u))\|\varphi'(u) du.$$

By the substitution $t = \varphi(u)$ we get $dt = \varphi'(u) du$ and

$$\int_c^d \|\tilde{\gamma}'(u)\|\varphi'(u) du = \int_{\varphi(c)}^{\varphi(d)} \|\gamma'(t)\| dt = \int_a^b \|\gamma'(t)\| dt.$$

³Parametrizations of C that are not related in this way might not yield the same notion of length: consider two parametrizations of a circle that go once and twice around.

If $\varphi' < 0$ then $a = \varphi(d)$, $b = \varphi(c)$, and $\|\tilde{\gamma}'(u)\| = \|\gamma'(\varphi(u))\|\varphi'(u)| = \|\gamma'(\varphi(u))\|(-\varphi'(u))$, so

$$\int_c^d \|\tilde{\gamma}'(u)\| du = - \int_c^d \|\gamma'(\varphi(u))\|\varphi'(u) du.$$

By the substitution $t = \varphi(u)$ we get $dt = \varphi'(u) du$ and

$$- \int_c^d \|\gamma'(\varphi(u))\|\varphi'(u) du = - \int_{\varphi(c)}^{\varphi(d)} \|\gamma'(t)\| dt = - \int_b^a \|\gamma'(t)\| dt = \int_a^b \|\gamma'(t)\| dt.$$

□