# STABLY FREE MODULES 

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## 1. Introduction

Let $R$ be a commutative ring. When an $R$-module has a particular module-theoretic property after direct summing it with a finite free module, it is said to have the property stably. For example, $R$-modules $M$ and $N$ are stably isomorphic if $R^{k} \oplus M \cong R^{k} \oplus N$ for some $k \geq 0$. An $R$-module $M$ is stably free if it is stably isomorphic to a free module: $R^{k} \oplus M$ is free for some $k$. When $M$ is finitely generated and stably free, then for some $k$ $R^{k} \oplus M$ is finitely generated and free, so $R^{k} \oplus M \cong R^{\ell}$ for some $\ell$. Necessarily $k \leq \ell$ (why?). Are stably isomorphic modules in fact isomorphic? Is a stably free module actually free? Not always, and that's why the concepts are interesting. This "stable mathematics" is part of algebraic $K$-theory. Our purpose here is to describe the simplest example of a non-free module that is stably free and then discuss what it means for all stably free modules over a ring to be free. It is due to Hochster (second paragraph of [2]).

Theorem 1.1. Let $R$ be the ring $\mathbf{R}[x, y, z] /\left(x^{2}+y^{2}+z^{2}-1\right)$. Let $T=\left\{(f, g, h) \in R^{3}\right.$ : $x f+y g+z h=0$ in $R\}$. Then $R \oplus T \cong R^{3}$, but $T \not \approx R^{2}$.

The module $T$ in this theorem is stably free (it is stably isomorphic to $R^{2}$ ), but it is not a free module. Indeed, if $T$ is free then (since $T$ is finitely generated; the theorem shows it admits a surjection from $R^{3}$ ) for some $n$ we have $T \cong R^{n}$, so $R \oplus R^{n} \cong R^{3}$. Since $R^{a} \cong R^{b}$ only if $a=b$ for nonzero commutative rings $R^{1} 1+n=3$ so $n=2$. But this contradicts the non-isomorphism in the conclusion of the theorem.

It's worth noting that the ranks in the theorem are as small as possible for a non-free stably free module. If $R$ is a commutative ring and $M$ is an $R$-module such that $R \oplus M \cong R$ then $M=0$. If $R \oplus M \cong R^{2}$ then $M \cong R$. The first time we could have $R \oplus M \cong R^{\ell}$ with $M \not \equiv R^{\ell-1}$ is $\ell=3$, and Theorem 1.1 shows such an example occurs.

## 2. Proof of Theorem 1.1

In the proof of Theorem 1.1 it will be easy to show $R \oplus T \cong R^{3}$. But the proof that $T \nVdash R^{2}$ will require a theorem from topology about vector fields on the sphere. We denote the module as $T$ because it is related to tangent vectors on the sphere.

Proof. Since $R$ is a ring, on $R^{3}$ we can consider the dot product $R^{3} \times R^{3} \rightarrow R$. For example, $(x, y, z) \cdot(x, y, z)=x^{2}+y^{2}+z^{2}=1$. For all $\mathbf{v} \in R^{3}$, let $r=\mathbf{v} \cdot(x, y, z) \in R$. Then

$$
(\mathbf{v}-r(x, y, z)) \cdot(x, y, z)=\mathbf{v} \cdot(x, y, z)-r(x, y, z) \cdot(x, y, z)=r-r=0
$$

[^0]so $\mathbf{v}-r(x, y, z) \in T$. That means $R^{3}=R(x, y, z)+T$. This sum is direct since $R(x, y, z) \cap$ $T=(0,0,0)$ : if $r(x, y, z) \in T$ then dotting $r(x, y, z)$ with $(x, y, z)$ implies $r=0$. So we have proved
\[

$$
\begin{equation*}
R^{3}=R(x, y, z) \oplus T \tag{2.1}
\end{equation*}
$$

\]

Since $R \cong R(x, y, z)$ by $r \mapsto r(x, y, z), R^{3} \cong R \oplus T$. Thus $T$ is stably free.
Now we will show by contradiction that $T \not \approx R^{2}$. Assume $T \cong R^{2}$, so $T$ has an $R$-basis of size 2 , say $(f, g, h)$ and $(F, G, H)$. By (2.1) the three vectors $(x, y, z),(f, g, h),(F, G, H)$ in $R^{3}$ are an $R$-basis, so the matrix

$$
\left(\begin{array}{lll}
x & f & F \\
y & g & G \\
z & h & H
\end{array}\right)
$$

in $\mathrm{M}_{3}(R)$ must be invertible: it is the change-of-basis matrix between the standard basis of $R^{3}$ and the basis $(x, y, z),(f, g, h),(F, G, H)$. Therefore the determinant of this matrix is a unit in $R$ :

$$
\operatorname{det}\left(\begin{array}{lll}
x & f & F  \tag{2.2}\\
y & g & G \\
z & h & H
\end{array}\right) \in R^{\times} .
$$

It makes sense to evaluate elements of $R$ at points $\left(x_{0}, y_{0}, z_{0}\right)$ on the unit sphere $S^{2}$ : polynomials in $\mathbf{R}[x, y, z]$ that are congruent modulo $x^{2}+y^{2}+z^{2}-1$ take the same value at all $\left(x_{0}, y_{0}, z_{0}\right) \in S^{2}$ since $x_{0}^{2}+y_{0}^{2}+z_{0}^{2}-1=0$. A unit in $R$ takes nonzero values everywhere on the sphere: if $a(x, y, z) b(x, y, z)=1$ in $R$ then $a\left(x_{0}, y_{0}, z_{0}\right) b\left(x_{0}, y_{0}, z_{0}\right)=1$ in $\mathbf{R}$ when $\left(x_{0}, y_{0}, z_{0}\right) \in S^{2}$. In particular, at each point $\mathbf{v} \in S^{2}$ the determinant in (2.2) has a nonzero value, so $(f(\mathbf{v}), g(\mathbf{v}), h(\mathbf{v})) \in \mathbf{R}^{3}-\{\mathbf{0}\}$. Thus $\mathbf{v} \mapsto(f(\mathbf{v}), g(\mathbf{v}), h(\mathbf{v}))$ is a nowhere vanishing vector field on $S^{2}$ with continuous components (polynomial functions are continuous). But this is impossible: the hairy ball theorem in topology says every continuous vector field on the sphere vanishes at least once.

There is a stably free non-free module $T_{\mathbf{Z}}$ over $\mathbf{Z}[x, y, z] /\left(x^{2}+y^{2}+z^{2}-1\right)$. The construction is analogous to the previous one. Elements of $\mathbf{Z}[x, y, z] /\left(x^{2}+y^{2}+z^{2}-1\right)$ can be evaluated on the real sphere, and the proof that $T_{\mathbf{Z}}$ is not a free module uses evaluations of polynomials at points on the real sphere as before.

For each $d \geq 1$, every continuous vector field on the $2 d$-dimensional sphere $S^{2 d}$ vanishes somewhere, so over

$$
\begin{equation*}
R=\mathbf{R}\left[x_{1}, \ldots, x_{2 d+1}\right] /\left(x_{1}^{2}+\cdots+x_{2 d+1}^{2}-1\right) \tag{2.3}
\end{equation*}
$$

the tangent module $T=\left\{\left(f_{1}, \ldots, f_{2 d+1}\right) \in R^{2 d+1}: \sum x_{i} f_{i}=0\right.$ in $\left.R\right\}$ is stably free but not free: $R \oplus T \cong R^{2 d+1}$ but $T \not \not 二 R^{2 d}$.

## 3. When Stably Free Modules Must Be Free

For some rings $R$, all stably free finitely generated $R$-modules are free. This holds if $R$ is a field since all vector spaces are free (have bases). It also holds if $R$ is a PID: a stably free $R$-module is a submodule of a finite free $R$-module, and every submodule of a finite free module over a PID is a free module. A much more difficult example is when $R=k\left[X_{1}, \ldots, X_{n}\right]$, where $k$ is a field. (This is Serre's conjecture, proved independently by

Quillen and Suslin with $k$ allowed to be a PID, not just a field. ${ }^{2}$ ) In this section we show how the task of proving all stably free finitely generated modules over a particular ring $R$ are free can be formulated as a linear algebra problem over $R$. (It is shown in the appendix that over every nonzero commutative ring, a non-finitely generated module that is stably free must be free, so there is no loss of generality in focusing on finitely generated modules.)

To distinguish $n$-tuples ( $a_{1}, \ldots, a_{n}$ ) in $R^{n}$ from the ideal $\left(a_{1}, \ldots, a_{n}\right)=R a_{1}+\cdots+R a_{n}$ in $R$, denote the $n$-tuple in $R^{n}$ as $\left[a_{1}, \ldots, a_{n}\right]$.

Theorem 3.1. Fix a nonzero commutative ring $R$ and a positive integer $n$. The following conditions are equivalent.
(1) For every $R$-module $M$, if $M \oplus R \cong R^{n}$ then $M$ is free.
(2) Every vector $\left[a_{1}, \ldots, a_{n}\right] \in R^{n}$ satisfying $\left(a_{1}, \ldots, a_{n}\right)=R$ is part of a basis of $R^{n}$.

Proof. Both (1) and (2) are true (for all $R$ ) when $n=1$, so we may suppose $n \geq 2$.
$(1) \Rightarrow(2)$ : Suppose $\left(a_{1}, \ldots, a_{n}\right)=R$, so $\sum a_{i} b_{i}=1$ for some $b_{i} \in R$. Set $\mathbf{a}=\left[a_{1}, \ldots, a_{n}\right]$ and $\mathbf{b}=\left[b_{1}, \ldots, b_{n}\right]$. Let $f: R^{n} \rightarrow R$ by $f(\mathbf{v})=\mathbf{v} \cdot \mathbf{b}$, so $f(\mathbf{a})=1$ and $R^{n}=R \mathbf{a} \oplus \operatorname{ker} f$ by the decomposition

$$
\mathbf{v}=f(\mathbf{v}) \mathbf{a}+(\mathbf{v}-f(\mathbf{v}) \mathbf{a}) .
$$

(This sum decomposition is unique because if $\mathbf{v}=r \mathbf{a}+\mathbf{w}$ with $r \in R$ and $\mathbf{w} \in \operatorname{ker} f$ then applying $f$ to both sides shows $f(\mathbf{v})=r$, so $\mathbf{w}=\mathbf{v}-r \mathbf{a}=\mathbf{v}-f(\mathbf{v}) \mathbf{a}$.) Since $R \mathbf{a} \cong R$ by $\mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{b}$ (concretely, $r \mathbf{a} \mapsto r$ ), $R^{n}$ is isomorphic to $R \oplus \operatorname{ker} f$, so $\operatorname{ker} f$ is free by (1). Adjoining a to a basis of ker $f$ provides us with a basis of $R^{n}$.
$(2) \Rightarrow(1):$ Let $g: M \oplus R \rightarrow R^{n}$ be an $R$-module isomorphism. Set $\mathbf{a}=g(0,1)=$ $\left[a_{1}, \ldots, a_{n}\right]$. To show the ideal $\left(a_{1}, \ldots, a_{n}\right)$ is $R$, suppose it is not. Then there is a maximal ideal $\mathfrak{m}$ containing each $a_{i}$, so $g(0,1) \subset \mathfrak{m}^{n}$. However, the isomorphism $g$ restricts to an isomorphism from $\mathfrak{m}(M \oplus R)=\mathfrak{m} M \oplus \mathfrak{m}$ to $\mathfrak{m} R^{n}=\mathfrak{m}^{n}$, so $g(0,1)$ being in $\mathfrak{m}^{n}$ implies $(0,1) \in \mathfrak{m} M \oplus \mathfrak{m}$, which is false.

By (2) there is a basis of $R^{n}$ containing a. Every $R$-basis of $R^{n}$ contains $n$ elements ${ }^{3}$ so we can write the basis of $R^{n}$ as $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ with $\mathbf{v}_{1}=\mathbf{a}$. Then $g^{-1}\left(\mathbf{v}_{1}\right), \ldots, g^{-1}\left(\mathbf{v}_{n}\right)$ is a basis of $M \oplus R$, with $g^{-1}\left(\mathbf{v}_{1}\right)=(0,1)$. For $i=2, \ldots, n$, write $g^{-1}\left(\mathbf{v}_{i}\right)=\left(m_{i}, c_{i}\right)$. Subtracting a multiple of $(0,1)$ from each $\left(m_{i}, c_{i}\right)$ for $i=2, \ldots, n$, we get a basis $(0,1),\left(m_{2}, 0\right), \ldots,\left(m_{n}, 0\right)$ of $M \oplus R$. Writing ( $m, 0$ ) in $M \oplus R$ as a linear combination of these shows $m_{2}, \ldots, m_{n}$ spans $M$ as an $R$-module and is linearly independent, so $M$ is free.

Corollary 3.2. For a commutative ring $R$, the following conditions are equivalent.
(1) For all $R$-modules $M$, if $M \oplus R \cong R^{n}$ for some $n$ then $M$ is free.
(2) For all $n \geq 1$, every vector $\left[a_{1}, \ldots, a_{n}\right] \in R^{n}$ satisfying $\left(a_{1}, \ldots, a_{n}\right)=R$ is part of a basis of $R^{n}$.
(3) All stably free finitely generated $R$-modules are free.

Proof. (1) $\Leftrightarrow(2)$ : This equivalence is Theorem 3.1 for all $n$.
$(1) \Rightarrow(3)$ : Suppose $M$ is a stably free $R$-module, so $M \oplus R^{k} \cong R^{\ell}$ for some $k$ and $\ell$. We want to show $M$ is free. If $k=0$ then obviously $M$ is free. If $k \geq 1$ then $\left(M \oplus R^{k-1}\right) \oplus R \cong R^{\ell}$, so (1) with $n=\ell$ tells us that $M \oplus R^{k-1}$ is free. By induction on $k$, the module $M$ is free.

[^1](3) $\Rightarrow$ (1): If $M \oplus R \cong R^{n}$ for some $n$ then $M$ is stably free, and thus $M$ is free by (3).

Corollary $3.2(2)$ expresses the freeness of all stably free finitely generated $R$-modules as a problem in linear algebra in $R^{n}$ (over all $n$ ). The condition there that the coordinates generate the unit ideal is necessary if $\left[a_{1}, \ldots, a_{n}\right]$ has a chance to be part of a basis of $R^{n}$ :
Theorem 3.3. If $\left[a_{1}, \ldots, a_{n}\right] \in R^{n}$ is part of a basis of $R^{n}$ then the ideal $\left(a_{1}, \ldots, a_{n}\right)$ is the unit ideal.

Proof. We are assuming there is an $R$-basis of $R^{n}$ that contains the $n$-tuple $\left[a_{1}, \ldots, a_{n}\right]$. Any basis has $n$ elements, so write the basis as $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ with $\mathbf{v}_{1}=\left[a_{1}, \ldots, a_{n}\right]$. Write each $\mathbf{v}_{j}$ in coordinates relative to the standard basis of $R^{n}$, say $\mathbf{v}_{j}=\left[c_{1 j}, \ldots, c_{n j}\right]$ (so $a_{i}=c_{i 1}$ ). Then the matrix ( $c_{i j}$ ) has the $\mathbf{v}_{j}$ 's as its columns, so this matrix describes the linear transformation $R^{n} \rightarrow R^{n}$ sending the standard basis of $R^{n}$ to $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$. Since the $\mathbf{v}_{j}$ 's form a basis, this matrix is invertible: $\operatorname{det}\left(c_{i j}\right) \in R^{\times}$. Expanding the determinant along its first column shows $\operatorname{det}\left(c_{i j}\right)$ is an $R$-linear combination of $a_{1}, \ldots, a_{n}$, so $\operatorname{det}\left(c_{i j}\right) \in\left(a_{1}, \ldots, a_{n}\right)$. Therefore the ideal $\left(a_{1}, \ldots, a_{n}\right)$ contains a unit, so the ideal is $R$.

Thus all stably free finitely generated $R$-modules are free if and only if for all $n$ the "obvious" necessary condition for a vector in $R^{n}$ to be part of a basis of $R^{n}$ is a sufficient condition.

Example 3.4. If $\left[a_{1}, \ldots, a_{n}\right]$ in $\mathbf{Z}^{n}$ is part of a basis of $\mathbf{Z}^{n}$ then $\operatorname{gcd}\left(a_{1}, \ldots, a_{n}\right)=1$. For example, the vector $[6,9,15]$ is not part of a basis of $\mathbf{Z}^{3}$ since its coordinates are all multiples of 3. The vector $[6,10,15]$ has no common factors among its coordinates (although each pair of coordinates has a common factor). Is it part of a basis of $\mathbf{Z}^{3}$ ? Essentially we are asking if the necessary condition in Theorem 3.3 is also sufficient over $\mathbf{Z}$. It is in this case: the vectors $[6,10,15],[1,1,0]$, and $[0,3,11]$ are a basis of $\mathbf{Z}^{3}$. (A matrix with these vectors as the columns has determinant $\pm 1$.)

For a nonzero commutative ring $R$, the necessary condition $\left(a_{1}, \ldots, a_{n}\right)=R$ in Theorem 3.3 is actually sufficient for $\left[a_{1}, \ldots, a_{n}\right]$ to be part of a basis of $R^{n}$ when $n=1$ and $n=2$. For $n=1$, if $\left(a_{1}\right)=R$ then $a_{1}$ is a unit and thus is a basis of $R$ as an $R$-module. For $n=2$, if ( $a_{1}, a_{2}$ ) $=R$ then there are $b_{1}, b_{2} \in R$ such that $a_{1} b_{1}+a_{2} b_{2}=1$, so the matrix $\left(\begin{array}{cc}a_{1} & -b_{2} \\ a_{2} & b_{1}\end{array}\right)$ has determinant 1 and therefore its columns are a basis of $R^{2}$. What if $n>2$ ? The necessary condition is sufficient when $R$ is a PID by Corollary 3.2 since we already saw that stably free $\Rightarrow$ free when $R$ is a PID. (e.g., $R=\mathbf{Z}$ or $F[X])$. More generally, the necessary condition is sufficient when $R$ is a Dedekind domain [7], [8], but Theorem 1.1 provides us with a ring admitting a stably free module that is not free, and this leads to a counterexample when $n=3$.
Example 3.5. Let $R=\mathbf{R}[x, y, z] /\left(x^{2}+y^{2}+z^{2}-1\right)$. In the free module $R^{3}$, the triple $[x, y, z]$ satisfies the condition of Theorem 3.3: the ideal $(x, y, z)$ of $R$ is the unit ideal since $x^{2}+y^{2}+z^{2}=1$ in $R$. However, there is no basis of $R^{3}$ containing $[x, y, z]$. Indeed, assume there is a basis $\mathbf{v}_{1}, \mathbf{v}_{2}, \mathbf{v}_{3}$ where $\mathbf{v}_{1}=[x, y, z]$. Then there is a matrix in $\mathrm{GL}_{3}(R)$ with first column $\mathbf{v}_{1}$, and the argument in the proof of Theorem 1.1 derives a contradiction from this.

The "sphere rings" $\mathbf{R}\left[x_{1}, \ldots, x_{n}\right] /\left(x_{1}^{2}+\cdots+x_{n}^{2}-1\right)$ for odd $n \geq 3$ provide additional examples where the condition in Theorem 3.3 is not sufficient to guarantee an $n$-tuple in $R^{n}$ is part of a basis of $R^{n}$.

Another use of the ring $R=\mathbf{R}[x, y, z] /\left(x^{2}+y^{2}+z^{2}-1\right)$ as a counterexample in algebra involves matrices with trace 0 . Since $\operatorname{Tr}(A B)=\operatorname{Tr}(B A)$, every matrix of the form $A B-B A$ (called a commutator) has trace 0 . Over a field, the converse holds [1]: every square matrix with trace 0 is a commutator. However, the matrix $\left(\begin{array}{cc}x & y \\ z & -x\end{array}\right)$ in $\mathrm{M}_{2}(R)$ has trace 0 and it is proved in [9] that this matrix is not a commutator in $\mathrm{M}_{2}(R)$. But all is not lost. A matrix with trace 0 is always a sum of two commutators [6].

## Appendix A. Theorems of Gabel and Bass

Our discussion of stably free modules focused on finitely generated ones. The reason is that in the case of non-finitely generated modules there are no interesting stably free modules.

Theorem A. 1 (Gabel). If $M$ is stably free and not finitely generated then $M$ is free.
Proof. Let $F:=M \oplus R^{k}$ be free. We want to show $M$ is free.
Projection from $F$ onto $M$ is a surjective linear map, so $M$ not being finitely generated implies $F$ is not finitely generated. Let $\left\{e_{i}\right\}_{i \in I}$ be a basis of $F$, so the index set $I$ is infinite.

Projection from $F$ onto $R^{k}$ is a surjective linear map $f: F \rightarrow R^{k}$ with kernel $M$. The standard basis of $R^{k}$ is in the image of the span of finitely many $e_{i}$ 's, say the submodule $F^{\prime}:=R e_{1}+\cdots+R e_{\ell}$ has $f\left(F^{\prime}\right)=R^{k}$. For each $\mathbf{v} \in F, f(\mathbf{v})=f\left(\mathbf{v}^{\prime}\right)$ for some $\mathbf{v}^{\prime} \in F^{\prime}$. Then $\mathbf{v}-\mathbf{v}^{\prime} \in \operatorname{ker} f=M$, so $F=F^{\prime}+M$. The module $F^{\prime}$ is finite free and $F^{\prime} /\left(M \cap F^{\prime}\right) \cong R^{k}$. Since $R^{k}$ is free (and thus a projective module), there is an isomorphism $F^{\prime} \cong N \oplus R^{k}$ where $N=M \cap F^{\prime}$. Since $F^{\prime}+M=F, F / F^{\prime} \cong M / N$ and $F / F^{\prime}=\bigoplus_{i>\ell} R e_{i}$ is free with infinite rank, so we can write $F / F^{\prime} \cong R^{k} \oplus F^{\prime \prime}$ for some free $F^{\prime \prime}$. Therefore $M / N$ is free, so

$$
M \cong N \oplus(M / N) \cong N \oplus\left(F / F^{\prime}\right) \cong N \oplus R^{k} \oplus F^{\prime \prime} \cong F^{\prime} \oplus F^{\prime \prime}
$$

which is free.
To prove all stably free modules over a (nonzero commutative) ring $R$ are free is the same as showing $M \oplus R^{k} \cong R^{\ell} \Rightarrow M$ is free for all $k$ and $\ell$. When such an isomorphism occurs, $\ell-k=\operatorname{dim}_{R / \mathfrak{m}}(M / \mathfrak{m} M)$ for all maximal ideals $\mathfrak{m}$ in $R$, so $\ell-k$ is well-defined by $M$ although $\ell$ and $k$ are not. We call $\ell-k$ the rank of $M$. For example, if $M \oplus R \cong R^{n}$ then $M$ has rank $n-1$. We will prove a theorem of Bass that reduces the verification that all stably free $R$-modules are free to the case of even rank.

Lemma A.2. If $M \oplus R \cong R^{2 d}$ for some $d \geq 1$ then $M \cong R \oplus N$ for some $R$-module $N$.
Proof. Composing an isomorphism $R^{2 d} \cong M \oplus R$ with projection to the second summand gives us a surjective map $\varphi: R^{2 d} \rightarrow R$ with kernel isomorphic to $M$. Since every linear map $R^{2 d} \rightarrow R$ is dotting with a fixed vector, there is some $\mathbf{w} \in R^{2 d}$ such that $\varphi(\mathbf{v})=\mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{v} \in R^{2 d}$. Set $\mathbf{w}=\left(c_{1}, \ldots, c_{2 d}\right)$. Then

$$
\varphi\left(c_{2},-c_{1}, \ldots, c_{2 d},-c_{2 d-1}\right)=0
$$

Let $\mathbf{u}=\left(c_{2},-c_{1}, \ldots, c_{2 d},-c_{2 d-1}\right) \in \operatorname{ker} \varphi \cong M$. We will show there is a submodule $N$ of $M$ such that $M \cong R \oplus N$.

Choose $\left(r_{1}, \ldots, r_{2 d}\right) \in R^{2 d}$ such that $\varphi\left(r_{1}, \ldots, r_{2 d}\right)=1$, so $c_{1} r_{1}+\cdots+c_{2 d} r_{2 d}=1$. Then $\mathbf{u} \cdot\left(r_{2},-r_{1}, \ldots, r_{2 d},-r_{2 d-1}\right)=1$, so the linear map $f: R^{2 d} \rightarrow R$ given by $f(\mathbf{v})=$ $\mathbf{v} \cdot\left(r_{2},-r_{1}, \ldots, r_{2 d},-r_{2 d-1}\right)$ satisfies $f(\mathbf{u})=1$. Since $\mathbf{u} \in \operatorname{ker} \varphi$, the restriction of $f$ to a linear map $\operatorname{ker} \varphi \rightarrow R$ is surjective and restricts to an isomorphism $R \mathbf{u} \rightarrow R$. Thus $M \cong \operatorname{ker} \varphi=R \mathbf{u} \oplus \operatorname{ker} f \cong R \oplus \operatorname{ker} f$.

Remark A.3. It is generally false that if $M \oplus R \cong R^{2 d+1}$ then $M \cong R \oplus N$ for some $N$. An example is $R$ being the sphere ring (2.3) when $2 d+1 \neq 1,3$ or 7 and $M$ being the tangent module $T$. Our work in Section 2 shows $T \oplus R \cong R^{2 d+1}$. A proof that $T \not \not \approx R \oplus N$ for an $R$-module $N$ is in [4, pp. 33-35].
Theorem A. 4 (Bass). The following conditions on a commutative ring $R$ are equivalent.
(1) All stably free finitely generated $R$-modules are free.
(2) All stably free finitely generated $R$-modules of even rank are free.

Proof. It's clear that (1) $\Rightarrow(2)$. To show (2) $\Rightarrow(1)$, suppose $M \oplus R^{k} \cong R^{\ell}$ (so $k \leq \ell$ ) with $\ell-k$ an odd number. If $k=0$ then $M$ is free. If $k>0$ then $(M \oplus R) \oplus R^{k-1} \cong R^{\ell}$, so $M \oplus R$ is stably free of even rank $\ell-(k-1)$. Then $M \oplus R \cong R^{\ell-k+1}$, so $M \cong R \oplus N$ for some $N$ by Lemma A.2. Therefore $N \oplus R^{2}$ is free of even rank $\ell-k+1$, so $N$ is stably free of odd $\operatorname{rank}(\ell-k+1)-2=\ell-k-1$. By induction $N \cong R^{\ell-k-1}$, so $M \cong R \oplus N \cong R^{\ell-k}$.

## References

[1] A. A. Albert and B. Muckenhoupt, On matrices of trace zero, Michigan Math. J. 4 (1957), 1-3.
[2] M. Hochster, Nonuniqueness of coefficient rings in a polynomial ring, Proc. Amer. Math. Soc. 34 (1972), 81-82.
[3] W.H. Gustafson, P. R. Halmos, J. M. Zelmanowitz, The Serre Conjecture, Amer. Math. Monthly 85 (1978), 357-359.
[4] T. Y. Lam, "Serre's Problem on Projective Modules," Springer-Verlag, Berlin, 1978.
[5] B. McDonald, "Linear Algebra over Commutative Rings," Marcel Dekker, New York, 1984.
[6] Z. Mesyan, Commutator Rings, Bull Austral. Math Soc. 74 (2006), 279-288.
[7] I. Reiner, Unimodular Complements, Amer. Math. Monthly 63 (1956), 246-247.
[8] I. Reiner, Completion of Primitive Matrices, Amer. Math. Monthly 73 (1966), 380-381.
[9] M. Rosset and S. Rosset, Elements of trace zero that are not commutators, Comm. Algebra 28 (2000), 3059-3072.


[^0]:    ${ }^{1}$ If $R^{a} \cong R^{b}$ as $R$-modules then, for a maximal ideal $\mathfrak{m}$ in $R$, that $R$-module isomorphism implies $\mathfrak{m}^{a} \cong \mathfrak{m}^{b}$ and therefore $R^{a} / \mathfrak{m}^{a} \cong R^{b} / \mathfrak{m}^{b}$, so $(R / \mathfrak{m})^{a} \cong(R / \mathfrak{m})^{b}$ as $R / \mathfrak{m}$-vector spaces. By linear algebra over the field $R / \mathfrak{m}, a=b$.

[^1]:    ${ }^{2}$ The actual problem put forward by Serre was to show every finitely generated projective module over $k\left[X_{1}, \ldots, X_{n}\right]$ is free. He showed such modules are stably free, so his problem reduces to the version we stated about freeness of stably free finitely generated modules over $k\left[X_{1}, \ldots, X_{n}\right]$.
    ${ }^{3}$ If a basis contains $n^{\prime}$ elements then $R^{n} \cong R^{n^{\prime}}$, so $n^{\prime}=n$ by the first footnote.

