BASE EXTENSION AND EXTERIOR POWERS

KEITH CONRAD

Let $f\colon R \to S$ be a ring homomorphism, through which we can base extend R-modules to S-modules: $M \rightsquigarrow S \otimes_R M$ and $\Lambda^k(M) \rightsquigarrow S \otimes_R \Lambda^k(M)$. We will write $\Lambda^k(M)$ as $\Lambda^k_R(M)$ to emphasize the dependence on the scalar ring, since we will be working with exterior powers of both R-modules and S-modules.

Theorem 1. *Base extension commutes with formation of exterior powers: for any R-module M and integer $k \geq 1$, there is a unique S-module isomorphism $S \otimes_R \Lambda^k_R(M) \cong \Lambda^k_S(S \otimes_R M)$ where*

$$s \otimes (m_1 \wedge_R \cdots \wedge_R m_k) \mapsto s((1 \otimes m_1) \wedge_S \cdots \wedge_S (1 \otimes m_k)).$$

Moreover, if $M \xrightarrow{\varphi} N$ is a linear map of R-modules then the diagram

$$\begin{tikzcd}
S \otimes_R \Lambda^k_R(M) \arrow[r, "1\otimes\wedge^k(\varphi)"] \arrow[d] & S \otimes_R \Lambda^k_R(N) \arrow[d] \\
\Lambda^k_S(S \otimes_R M) \arrow[r, "\wedge^k(1\otimes\varphi)"'] & \Lambda^k_S(S \otimes_R N)
\end{tikzcd}$$

commutes.

For $k = 0$ the S-modules $S \otimes_R \Lambda^0_R(M) = S \otimes_R R$ and $\Lambda^0_S(S \otimes_R M) = S$ are naturally isomorphic, but the formula in Theorem 1 makes no sense for $k = 0$ so we omitted that case from the theorem. If you consider an empty wedge product to be 1, then that formula does make sense when $k = 0$ and is an isomorphism ($s \otimes 1 \mapsto s$).

Proof. For $k = 1$ the indicated function is $S \otimes_R M \to S \otimes_R M$ by $s \otimes m \mapsto s(1 \otimes m) = s \otimes m$, so it is just the identity map. From now on let $k \geq 2$. We want to write down S-linear maps $S \otimes_R \Lambda^k_R(M) \to \Lambda^k_S(S \otimes_R M)$ and $S \otimes_R \Lambda^k_R(M) \to \Lambda^k_S(S \otimes_R M)$ that are inverses of each other.

To write down an S-linear map

$$S \otimes_R \Lambda^k_R(M) \to \Lambda^k_S(S \otimes_R M),$$

we first ignore the S-part on the left and concentrate on getting an R-linear map

$$\Lambda^k_R(M) \to \Lambda^k_S(S \otimes_R M), \tag{1}$$

which we will then turn into an S-linear map out of $S \otimes_R \Lambda^k_R(M)$ using extension of scalars. The function we want to use is this:

$$m_1 \wedge_R \cdots \wedge_R m_k \mapsto (1 \otimes_R m_1) \wedge_S \cdots \wedge_S (1 \otimes_R m_k). \tag{2}$$

Strictly speaking, the rule (2) on its own does not *define* an R-linear map out of $\Lambda^k_R(M)$ because the elements of $\Lambda^k_R(M)$ are not all elementary wedge products, although there is at most one R-linear map with the effect (2) since elementary wedge products additively span

$\Lambda^k_R(M)$. To construct an R-linear map (1), we write down an alternating R-multilinear map out of M^k. Define $M^k \to \Lambda^k_S(S \otimes_R M)$ by

$$(m_1, \ldots, m_k) \mapsto (1 \otimes_R m_1) \wedge_S \cdots \wedge_S (1 \otimes_R m_k).$$

This function is alternating and R-multilinear (treat the target $\Lambda^k_S(S \otimes_R M)$ as an R-module in the way every S-module is an R-module: restriction of scalars). Therefore the universal mapping property of $\Lambda^k_R(M)$ gives us an R-linear map $\Lambda^k_R(M) \to \Lambda^k_S(S \otimes_R M)$ satisfying (2) on elementary wedge products.

Now we use the universal mapping property of extension of scalars: from the R-linear map $\Lambda^k_R(M) \to \Lambda^k_S(S \otimes_R M)$ in (1) there is an S-linear map $\varphi\colon S \otimes_R \Lambda^k_R(M) \to \Lambda^k_S(S \otimes_R M)$ such that

$$\varphi(s \otimes_R (m_1 \wedge_R \cdots \wedge_R m_k)) = s((1 \otimes_R m_1) \wedge_S \cdots \wedge_S (1 \otimes_R m_k)). \tag{3}$$

This is the formula we want in the theorem.

To get an S-linear map in the other direction, $\Lambda^k_S(S \otimes_R M) \to S \otimes_R \Lambda^k_R(M)$, we want to write down an appropriate alternating S-multilinear map $(S \otimes_R M)^k \to S \otimes_R \Lambda^k_R(M)$ and then use the universal mapping property of $\Lambda^k_S(S \otimes_R M)$. Consider the composite map

$$(S \otimes_R M)^k \xrightarrow{\otimes_S} (S \otimes_R M)^{\otimes_S k} \cong S \otimes_R (M^{\otimes_R k}) \xrightarrow{1 \otimes \wedge_R} S \otimes_R \Lambda^k_R(M). \tag{4}$$

The first map is the canonical multilinear map defining tensor powers (of S-modules), the isomorphism in the middle is the inverse of the canonical one for base extensions of tensor products (recall for two R-modules M and N that $S \otimes_R (M \otimes_R N) \cong (S \otimes_R M) \otimes_S (S \otimes_R N)$ by $s \otimes (m \otimes n) \mapsto s(1 \otimes m) \otimes (1 \otimes n)$), and the third map is the base extension to S-modules of the canonical R-linear map $\wedge_R\colon M^{\otimes_R k} \to \Lambda^k_R(M)$. The first map is S-multilinear and the second and third are S-linear, so the composite map is S-multilinear. Its effect on a k-tuple of elementary tensors is

$$\begin{aligned}(s_1 \otimes_R m_1, \ldots, s_k \otimes_R m_k) &\mapsto (s_1 \otimes_R m_1) \otimes_S \cdots \otimes_S (s_k \otimes_R m_k)\\ &\mapsto s_1 \cdots s_k \otimes_R (m_1 \otimes_R \cdots \otimes_R m_k)\\ &\mapsto s_1 \cdots s_k \otimes_R (m_1 \wedge_R \cdots \wedge_R m_k).\end{aligned}$$

Let $\alpha\colon (S \otimes_R M)^k \to S \otimes_R \Lambda^k_R(M)$ denote the composite map (4). We want to show α is alternating: $\alpha(t_1, t_2, \ldots, t_k) = 0$ when $t_i = t_j$ for some $i \neq j$. Because α is multilinear, it suffices to check such vanishing occurs when two coordinates are equal tensors and the other coordinates are all elementary tensors. To keep the notation simple, we focus on the case $t_1 = t_2$ (other equal coordinates run in the same way):

$$\alpha(t, t, s_3 \otimes_R m_3, \ldots, s_k \otimes_R m_k) \stackrel{?}{=} 0.$$

Writing $t = s'_1 \otimes_R m'_1 + \cdots + s'_n \otimes_R m'_n$, we expand the left side as

$$\begin{aligned}\alpha(t, t, s_3 \otimes_R m_3, \ldots, s_k \otimes_R m_k) &= \sum_{i,j=1}^{n} \alpha(s'_i \otimes_R m'_i, s'_j \otimes_R m'_j, s_3 \otimes_R m_3, \ldots, s_k \otimes_R m_k)\\ &= \sum_{i,j=1}^{n} s'_i s'_j s_3 \cdots s_k \otimes_R (m'_i \wedge_R m'_j \wedge_R m_3 \wedge_R \cdots \wedge_R m_k).\end{aligned}$$

In this sum, a term where $i = j$ is 0 because m'_i is repeated in the elementary wedge product. For $i \neq j$, the two terms with i and j in both orders are negatives of each other and thus cancel when added, so $\alpha(t, t, s_3 \otimes_R m_3, \ldots, s_k \otimes_R m_k) = 0$.

Since α is alternating multilinear, there is an S-linear map $\psi\colon \Lambda_S^k(S\otimes_R M) \to S\otimes_R \Lambda_R^k(M)$ such that

$$\psi((s_1 \otimes m_1) \wedge_S \cdots \wedge_S (s_k \otimes m_k)) = \alpha(s_1 \otimes m_1, \cdots, s_k \otimes m_k) = s_1 \cdots s_k \otimes (m_1 \wedge_R \cdots \wedge_R m_k).$$

It is left to the reader to show φ and ψ are inverses of each other. $\square$

Example 2. For any ideal $\mathfrak{a}$, $R/\mathfrak{a} \otimes_R \Lambda_R^k(M) \cong \Lambda_{R/\mathfrak{a}}^k(R/\mathfrak{a} \otimes_R M)$ as $R/\mathfrak{a}$-modules, and for any prime ideal $\mathfrak{p}$, $R_\mathfrak{p} \otimes_R \Lambda_R^k(M) \cong \Lambda_{R_\mathfrak{p}}^k(R_\mathfrak{p} \otimes_R M)$ as $R_\mathfrak{p}$-modules.