

# INFINITE-DIMENSIONAL DUAL SPACES

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Let  $K$  be a field and  $V$  be a  $K$ -vector space. The dual space  $V^\vee$  is the set of all  $K$ -linear functions  $\varphi: V \rightarrow K$ . If  $\dim_K(V)$  is finite then  $\dim_K(V) = \dim_K(V^\vee)$ , so  $V$  and  $V^\vee$  are (non-canonically) isomorphic  $K$ -vector spaces and there is a canonical isomorphism  $V \rightarrow V^{\vee\vee}$  by  $v \mapsto [\varphi \mapsto \varphi(v)]$ . If  $\dim_K(V)$  is infinite, however, there is no isomorphism between  $V$  and  $V^{\vee\vee}$  because their dimensions are not equal by the following result.

**Theorem.** *If  $V$  is an infinite-dimensional  $K$ -vector space then  $\dim_K(V) < \dim_K(V^\vee)$ .*

Replacing  $V$  with  $V^\vee$  in this theorem, we get  $\dim_K(V) < \dim_K(V^\vee) < \dim_K(V^{\vee\vee})$  when  $V$  is infinite-dimensional over  $K$ , so  $V \not\cong V^{\vee\vee}$  as  $K$ -vector spaces.

This theorem uses the algebraic notion of dimension: the cardinality of a linearly independent spanning set where the linear combinations use *finitely many* nonzero vectors at a time. This is in contrast to analysis, where linear combinations may have infinitely many nonzero terms and the dual space is the *continuous* linear functionals. The fact that an infinite-dimensional Hilbert space (e.g.,  $\ell^2$  or  $L^2([0, 1])$ ) is isomorphic to its (continuous) dual space does not contradict the theorem above, since the dimension and dual space in analysis are smaller than in algebra.

The theorem above also depends on the axiom of choice: the existence of a basis for vector spaces that are infinite dimensional is in fact equivalent to the axiom of choice.

*Proof.* Our argument is based on an answer by Alcides Buss on MathOverflow [1].

**Step 1:**  $\dim_K(V^\vee)$  is infinite.

Pick a basis  $B = \{e_i\}_{i \in I}$  of  $V$ . Each element of  $V^\vee$  is determined by its values on  $B$  (then extend by linearity to all  $V$ ), and those values on  $B$  can be arbitrary.

For each  $i \in I$ , define  $\varphi_i \in V^\vee$  by setting  $\varphi_i(e_i) = 1$  and  $\varphi_i(e_j) = 0$  when  $j \neq i$ . Then  $\varphi_i(\sum_{j \in I} c_j e_j) = c_i$ , and the reader can use that to show  $\{\varphi_i\}_{i \in I}$  is a linearly independent subset of  $V^\vee$ . Thus  $V^\vee$  is infinite dimensional.

**Step 2:** If a  $K$ -vector space  $W$  is infinite dimensional, then

$$\text{card}(W) = \max(\text{card}(K), \dim_K(W)).$$

First we show  $\text{card}(K) \leq \text{card}(W)$  and  $\dim_K(W) \leq \text{card}(W)$ .

For nonzero  $w \in W$ , the set  $Kw$  is a subset of  $W$  with size  $\text{card}(K)$ , so  $\text{card}(K) \leq \text{card}(W)$ . A basis of  $W$  is a subset of  $W$  with size  $\dim_K(W)$ , so  $\dim_K(W) \leq \text{card}(W)$ .

Next we show  $\text{card}(W) \leq \max(\text{card}(K), \dim_K(W))$ . Pick a basis  $\{e_i\}_{i \in I}$  of  $W$ . The elements of  $w \in W$  are unique finite linear combinations  $\sum_{i \in I} c_i e_i$  where the  $c_i$  are in  $K$  and finitely many are nonzero, so we get an embedding of  $W$  into the *finite subsets* of  $K \times I$  by  $w \mapsto \{(c_i, e_i) : c_i \neq 0\}$ . (Note when  $w = 0$  we get the empty set in  $K \times I$ .) Since  $I$  is infinite,  $K \times I$  is infinite ( $K$  is nonempty), and the cardinality of the finite subsets of an infinite set equals the cardinality of the set. Thus  $\text{card}(W) \leq \text{card}(K \times I)$ . When  $A$  and  $B$  are

nonempty sets and at least one of them is infinite,  $\text{card}(A \times B) = \max(\text{card}(A), \text{card}(B))$ ,<sup>1</sup> so  $\text{card}(W) \leq \max(\text{card}(K), \text{card}(I)) = \max(\text{card}(K), \dim_K(W))$ .

**Step 3:**  $\text{card}(K) \leq \dim_K(V^\vee)$ .

We'll make an embedding  $K \rightarrow V^\vee$  where the image is linearly independent in  $V^\vee$ . Linearly independent subsets of  $V^\vee$  have size at most  $\dim_K(V^\vee)$ , so  $\text{card}(K) \leq \dim_K(V^\vee)$ .

Pick a basis  $\{e_i\}_{i \in I}$  of  $V$  and a countably infinite subset of it, say  $b_0, b_1, b_2, \dots$ , which is possible because  $V$  is infinite-dimensional. For each  $c \in K$ , define  $\varphi_c \in V^\vee$  by setting

$$\varphi_c(b_n) = c^n \text{ for } n \geq 0, \quad \varphi_c(e_i) = 0 \text{ if } e_i \notin \{b_0, b_1, b_2, \dots\}.$$

Note  $\varphi_c(b_0) = 1$  for all  $c$  (even  $c = 0$ ), so each  $\varphi_c$  is nonzero in  $V^\vee$ . And  $\varphi_c(b_1) = c$ , so different  $c$ 's lead to different  $\varphi_c$ 's. Therefore we have an injective function  $K \rightarrow V^\vee$  by  $c \mapsto \varphi_c$ . It remains to show the functionals  $\{\varphi_c\}_{c \in K}$  are linearly independent.

Suppose  $\sum_{r=1}^n a_r \varphi_{c_r} = 0$  in  $V^\vee$  for some  $n \geq 1$  and  $a_1, \dots, a_n$  in  $K$ . We want to show each  $a_r$  is 0. Evaluating the left side at  $b_0, b_1, \dots, b_r$ , we have the system of linear equations

$$\begin{aligned} a_1 + a_2 + \dots + a_n &= 0, \\ a_1 c_1 + a_2 c_2 + \dots + a_n c_n &= 0, \\ &\vdots \\ a_1 c_1^n + a_2 c_2^n + \dots + a_n c_n^n &= 0, \end{aligned}$$

so

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ c_1 & c_2 & \dots & c_n \\ & & \ddots & \\ c_1^n & c_2^n & \dots & c_n^n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The matrix on the left is a Vandermonde matrix, so its determinant is  $\prod_{j < k} (c_k - c_j)$ , which is nonzero since  $c_1, \dots, c_n$  are distinct. Thus  $a_1, a_2, \dots, a_n$  are all 0.

**Step 4:**  $\dim_K(V^\vee) = \text{card}(V^\vee)$ .

The dual space  $V^\vee$  is infinite-dimensional by Step 1. By Step 2 with  $W = V^\vee$  and Step 3,  $\text{card}(V^\vee) = \max(\text{card}(K), \dim_K(V^\vee)) = \dim_K(V^\vee)$ .

**Step 5:**  $\dim_K(V) < \dim_K(V^\vee)$ .

We will show  $\text{card}(V^\vee) > \dim_K(V)$ , so  $\dim_K(V^\vee) > \dim_K(V)$  by Step 4.

Let  $\{e_i\}_{i \in I}$  be a basis of  $V$ , so  $\dim_K(V) = \text{card}(I)$ . Elements of  $V^\vee$  are determined by their values on that basis, and those values can be arbitrary, so as a set  $V^\vee$  is in bijection with  $\prod_{i \in I} K$ .

Since  $|K| \geq 2$ ,  $\text{card}(V^\vee) \geq \text{card}(\prod_{i \in I} \{0, 1\})$ , and  $\prod_{i \in I} \{0, 1\}$  is in bijection with the power set  $\mathcal{P}(I)$ , the set of subsets of  $I$ . Thus  $\text{card}(V^\vee) \geq \text{card}(\mathcal{P}(I))$ . By Cantor's diagonal argument,  $\text{card}(\mathcal{P}(I)) > \text{card}(I)$  since  $I$  is infinite, so

$$\text{card}(V^\vee) \geq \text{card}(\mathcal{P}(I)) > \text{card}(I) = \dim_K(V). \quad \square$$

**Corollary.** *Let  $\dim_K(V)$  be infinite. For a subspace  $W$  of  $V$  let  $W^\perp = \{\varphi \in V^\vee : \varphi(w) = 0 \text{ for all } w \in W\}$  and for a subspace  $\Phi$  of  $V^\vee$ , let  $\Phi^\perp = \{v \in V : \varphi(v) = 0 \text{ for all } \varphi \in \Phi\}$ . Then  $W^\perp$  determines  $W$ , but there are  $\Phi$  such that  $\Phi^\perp$  does not determine  $\Phi$ .*

It's important that  $\dim_K(V)$  here is infinite. If it were finite then  $\Phi^\perp$  does determine  $\Phi$ .

<sup>1</sup>See <https://math.stackexchange.com/questions/979684>.

*Proof.* First we will show

$$W = \{v \in V : \varphi(v) = 0 \text{ for all } \varphi \in W^\perp\}.$$

The left side is contained in the right side by the definition of  $W^\perp$ . To prove the right side is contained in the left side, we'll show for each  $v \in V - W$  that there is  $\varphi \in W^\perp$  such that  $\varphi(v) \neq 0$ .

Let  $\{w_j\}_{j \in J}$  be a basis of  $W$ . Then  $v$  is linearly independent of  $\{w_j\}_{j \in J}$  since  $v \notin W$ , so  $V$  has a basis  $B$  containing  $\{w_j\}_{j \in J}$  and  $v$ . Define  $\varphi \in V^\vee$  by  $\varphi(b) = 0$  for all  $b \in B$  except  $b = v$ , and  $\varphi(v) = 1$ . Then  $\varphi(w_j) = 0$  for all  $j$ , so  $\varphi \in W^\perp$ , and  $\varphi(v) \neq 0$ .

Next we'll show there are many different subspaces  $\Phi$  of  $V^\vee$  for which  $\Phi^\perp = \{0\}$ . Let  $\{e_i\}_{i \in I}$  be a basis of  $V$ , and for each  $i_0 \in I$  let  $\varphi_{i_0} \in V^\vee$  be the  $e_{i_0}$ -coefficient function for this basis:  $\varphi_{i_0}(\sum x_i e_i) = x_{i_0}$ . For  $v \in V$  write  $v = \sum_{i \in I} x_i e_i$  (finitely many nonzero coefficients). If  $\varphi_i(v) = 0$  for all  $\varphi_i$  then  $x_i = 0$  for all  $i$ , so  $v = 0$ .

Let  $\Phi$  be the  $K$ -span of the functionals  $\varphi_i$  for  $i \in I$ , so the calculation we just made shows  $\Phi^\perp = \{0\}$ . Check the  $\varphi_i$ 's are  $K$ -linearly independent in  $V^\vee$ , so  $\dim_K \Phi = \text{card}(I) = \dim_K V < \dim_K V^\vee$ . Thus  $\Phi \neq V^\vee$ . If  $\Phi'$  is a subspace of  $V^\vee$  containing  $\Phi$  then  $\Phi'^\perp \subset \Phi^\perp = \{0\}$ , so  $\Phi'^\perp = \{0\}$ . There are infinitely many  $\Phi'$ : since  $\dim_K \Phi$  and  $\dim_K V^\vee$  are different infinite cardinal numbers, the quotient space  $V^\vee/\Phi$  is infinite-dimensional.  $\square$

#### REFERENCES

- [1] Alcides Buss, answer to MathOverflow question 13322, Slick proof?: A vector space has the same dimension as its dual if and only if it is finite dimensional, <http://mathoverflow.net/questions/13322>