INFINITE-DIMENSIONAL DUAL SPACES

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Let K be a field and V be a K-vector space. The dual space $V^\vee$ is the set of all K-linear functions $\varphi\colon V \to K$. If $\dim_K(V)$ is finite then $\dim_K(V) = \dim_K(V^\vee)$, so V and $V^\vee$ are (non-canonically) isomorphic K-vector spaces and there is a canonical isomorphism $V \to V^{\vee\vee}$ by $v \mapsto [\varphi \mapsto \varphi(v)]$. If $\dim_K(V)$ is infinite, however, there is no isomorphism between V and $V^{\vee\vee}$ because their dimensions are not equal by the following result.

Theorem 1. *If V is an infinite-dimensional K-vector space then* $\dim_K(V) < \dim_K(V^\vee)$ *as cardinal numbers.*

Replacing V with $V^\vee$ in Theorem 1, we get $\dim_K(V) < \dim_K(V^\vee) < \dim_K(V^{\vee\vee})$ when V is infinite-dimensional over K, so $V \not\cong V^{\vee\vee}$ as K-vector spaces.

This theorem uses the algebraic notion of dimension: the cardinality of a linearly independent spanning set where the linear combinations use *finitely many* nonzero vectors at a time. This is in contrast to analysis, where linear combinations may have infinitely many nonzero terms and the dual space is the *continuous* linear functionals. The fact that an infinite-dimensional Hilbert space (*e.g.*, ℓ^2 or $L^2([0,1])$) is isomorphic to its (continuous) dual space does not contradict the theorem above, since the dimension and dual space in analysis are smaller than in algebra.

Theorem 1 depends on the axiom of choice: the existence of a basis for vector spaces that are infinite dimensional is in fact equivalent to the axiom of choice.

Proof. Our argument is based on an answer by Alcides Buss on MathOverflow [1].

Step 1: $\dim_K(V^\vee)$ is infinite.

Pick a basis $B = \{e_i\}_{i\in I}$ of V. Each element of $V^\vee$ is determined by its values on B (then extend by linearity to all V), and those values on B can be arbitrary.

For each $i \in I$, define $\varphi_i \in V^\vee$ by setting $\varphi_i(e_i) = 1$ and $\varphi_i(e_j) = 0$ when $j \neq i$. Then $\varphi_i(\sum_{j\in I} c_j e_j) = c_i$, and the reader can use that to show $\{\varphi_i\}_{i\in I}$ is a linearly independent subset of $V^\vee$. Thus $V^\vee$ is infinite dimensional.

Step 2: If a K-vector space W is infinite dimensional, then

$$\operatorname{card}(W) = \max(\operatorname{card}(K), \dim_K(W)).$$

First we show $\operatorname{card}(K) \leq \operatorname{card}(W)$ and $\dim_K(W) \leq \operatorname{card}(W)$.

For nonzero $w \in W$, the set Kw is a subset of W with size $\operatorname{card}(K)$, so $\operatorname{card}(K) \leq \operatorname{card}(W)$. A basis of W is a subset of W with size $\dim_K(W)$, so $\dim_K(W) \leq \operatorname{card}(W)$.

Next we show $\operatorname{card}(W) \leq \max(\operatorname{card}(K), \dim_K(W))$. Pick a basis $\{e_i\}_{i\in I}$ of W. The elements of $w \in W$ are unique finite linear combinations $\sum_{i\in I} c_i e_i$ where the c_i are in K and finitely many are nonzero, so we get an embedding of W into the *finite subsets* of $K \times I$ by $w \mapsto \{(c_i, e_i) : c_i \neq 0\}$. (Note when $w = 0$ we get the empty set in $K \times I$.) Since I is infinite, $K \times I$ is infinite (K is nonempty), and the cardinality of the finite subsets of an infinite set equals the cardinality of the set. Thus $\operatorname{card}(W) \leq \operatorname{card}(K \times I)$. When A and B are nonempty sets and at least one of them is infinite, $\text{card}(A \times B) = \max(\text{card}(A), \text{card}(B))$,[1] so $\text{card}(W) \le \max(\text{card}(K), \text{card}(I)) = \max(\text{card}(K), \dim_K(W))$.

Step 3: $\text{card}(K) \le \dim_K(V^\vee)$.

We'll make an embedding $K \to V^\vee$ where the image is linearly independent in $V^\vee$. Linearly independent subsets of $V^\vee$ have size at most $\dim_K(V^\vee)$, so $\text{card}(K) \le \dim_K(V^\vee)$.

Pick a basis $\{e_i\}_{i \in I}$ of V and a countably infinite subset of it, say $b_0, b_1, b_2, \ldots$, which is possible because V is infinite-dimensional. For each $c \in K$, define $\varphi_c \in V^\vee$ by setting

$$\varphi_c(b_n) = c^n \text{ for } n \ge 0, \quad \varphi_c(e_i) = 0 \ \text{ if } e_i \notin \{b_0, b_1, b_2, \ldots\}.$$

Note $\varphi_c(b_0) = 1$ for all c (even $c = 0$), so each φ_c is nonzero in $V^\vee$. And $\varphi_c(b_1) = c$, so different c's lead to different φ_c's. Therefore we have an injective function $K \to V^\vee$ by $c \mapsto \varphi_c$. It remains to show the functionals $\{\varphi_c\}_{c \in K}$ are linearly independent.

Suppose $\sum_{r=1}^n a_r \varphi_{c_r} = 0$ in $V^\vee$ for some $n \ge 1$ and $a_1, \ldots, a_n$ in K. We want to show each a_r is 0. Evaluating the left side at $b_0, b_1, \ldots, b_r$, we have the system of linear equations

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &= 0, \\ a_1 c_1 + a_2 c_2 + \cdots + a_n c_n &= 0, \\ &\vdots \\ a_1 c_1^n + a_2 c_2^n + \cdots + a_n c_n^n &= 0, \end{aligned}$$

so

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_1^n & c_2^n & \cdots & c_n^n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The matrix on the left is a Vandermonde matrix, so its determinant is $\prod_{j<k}(c_k - c_j)$, which is nonzero since $c_1, \ldots, c_n$ are distinct. Thus $a_1, a_2, \ldots, a_n$ are all 0.

Step 4: $\dim_K(V^\vee) = \text{card}(V^\vee)$.

The dual space $V^\vee$ is infinite-dimensional by Step 1. By Step 2 with $W = V^\vee$ and Step 3, $\text{card}(V^\vee) = \max(\text{card}(K), \dim_K(V^\vee)) = \dim_K(V^\vee)$.

Step 5: $\dim_K(V) < \dim_K(V^\vee)$.

We will show $\text{card}(V^\vee) > \dim_K(V)$, so $\dim_K(V^\vee) > \dim_K(V)$ by Step 4.

Let $\{e_i\}_{i \in I}$ be a basis of V, so $\dim_K(V) = \text{card}(I)$. Elements of $V^\vee$ are determined by their values on that basis, and those values can be arbitrary, so as a set $V^\vee$ is in bijection with $\prod_{i \in I} K$.

Since $|K| \ge 2$, $\text{card}(V^\vee) \ge \text{card}(\prod_{i \in I}\{0,1\})$, and $\prod_{i \in I}\{0,1\}$ is in bijection with the power set $\mathcal{P}(I)$, the set of subsets of I. Thus $\text{card}(V^\vee) \ge \text{card}(\mathcal{P}(I))$. By Cantor's diagonal argument, $\text{card}(\mathcal{P}(I)) > \text{card}(I)$ since I is infinite, so

$$\text{card}(V^\vee) \ge \text{card}(\mathcal{P}(I)) > \text{card}(I) = \dim_K(V). \qquad \square$$

Corollary 2. *Let $\dim_K(V)$ be infinite. For a subspace W of V let*

$$W^\perp = \{\varphi \in V^\vee : \varphi(w) = 0 \text{ for all } w \in W\},$$

and for a subspace Φ of $V^\vee$, let

$$\Phi^\perp = \{v \in V : \varphi(v) = 0 \text{ for all } \varphi \in \Phi\}.$$

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[1] See https://math.stackexchange.com/questions/979684.

Then $W^\perp$ determines W, but there are subspaces Φ in $V^\vee$ such that $\Phi^\perp$ does not determine Φ.

It's important that $\dim_K(V)$ here is infinite. If it were finite then $\Phi^\perp$ does determine Φ.

Proof. First we will show

$$W = \{v \in V : \varphi(v) = 0 \text{ for all } \varphi \in W^\perp\}.$$

The left side is contained in the right side by the definition of $W^\perp$. To prove the right side is contained in the left side, we'll show for each $v \in V - W$ that there is $\varphi \in W^\perp$ such that $\varphi(v) \neq 0$.

Let $\{w_j\}_{j\in J}$ be a basis of W. Then v is linearly independent of $\{w_j\}_{j\in J}$ since $v \notin W$, so V has a basis B containing $\{w_j\}_{j\in J}$ and v. Define $\varphi \in V^\vee$ by $\varphi(b) = 0$ for all $b \in B$ except $b = v$, and $\varphi(v) = 1$. Then $\varphi(w_j) = 0$ for all j, so $\varphi \in W^\perp$, and $\varphi(v) \neq 0$.

Next we'll show there are many different subspaces Φ of $V^\vee$ for which $\Phi^\perp = \{0\}$. Let $\{e_i\}_{i\in I}$ be a basis of V, and for each $i_0 \in I$ let $\varphi_{i_0} \in V^\vee$ be the e_{i_0}-coefficient function for this basis: $\varphi_{i_0}(\sum x_i e_i) = x_{i_0}$. For $v \in V$ write $v = \sum_{i\in I} x_i e_i$ (finitely many nonzero coefficients). If $\varphi_i(v) = 0$ for all φ_i then $x_i = 0$ for all i, so $v = 0$.

Let Φ be the K-span of the functionals φ_i for $i \in I$, so the calculation we just made shows $\Phi^\perp = \{0\}$. Check the φ_i's are K-linearly independent in $V^\vee$, so $\dim_K \Phi = \text{card}(I) = \dim_K V < \dim_K V^\vee$. Thus $\Phi \neq V^\vee$. If Φ' is a subspace of $V^\vee$ containing Φ then $\Phi'^\perp \subset \Phi^\perp = \{0\}$, so $\Phi'^\perp = \{0\}$. There are infinitely many Φ': since $\dim_K \Phi$ and $\dim_K V^\vee$ are different infinite cardinal numbers, the quotient space $V^\vee/\Phi$ is infinite-dimensional. □

We will now give an application of Theorem 1 to the existence of finite-index non-open subgroups in certain topological groups. We will assume the reader has a basic familiarity with topological groups.

A standard property of topological groups is that all open subgroups are closed and all closed subgroups with finite index are open. In particular, a subgroup having finite index is open if and only if it is closed. This leaves open (pun intended) the question whether there is an example of a topological group with a subgroup of finite index that is *not* open, or equivalently is not closed. The next theorem give such an example.

Theorem 3. *For a prime p, make the countable product $G := \prod_{n\geq 1} \mathbf{F}_p$ into a topological group by giving each $\mathbf{F}_p$ the discrete topology and G the product topology. Then G has subgroups with index p that are not open.*

Proof. This proof is adapted from an answer by Lukas Heger on Math Stackexchange [2].

Each homomorphism $G \to \mathbf{F}_p$ has kernel G or a kernel with index p. When G has the product topology and $\mathbf{F}_p$ has the discrete topology, a continuous homomorphism $G \to \mathbf{F}_p$ has a closed kernel since both subgroups of $\mathbf{F}_p$ are closed in the discrete topology. Thus a homomorphism $G \to \mathbf{F}_p$ that is *continuous* has a closed kernel that is G or has index p. (Conversely, all closed subgroups of G with index p do arise as the kernel of a continuous homomorphism $G \to \mathbf{F}_p$, but we do not need this.)

To show G has an index-p subgroup that is not open, or equivalently not closed, we will show there is a homomorphism $G \to \mathbf{F}_p$ that is *not* continuous. We will use linear algebra: homomorphisms $G \to \mathbf{F}_p$ are the same thing as $\mathbf{F}_p$-linear maps.

Let D be the direct sum $\bigoplus_{n\geq 1} \mathbf{F}_p$: this is an $\mathbf{F}_p$-vector space with a countable basis $e_n = (\ldots, 0, 0, 1, 0, 0, \ldots)$. Each element in $D^\vee$, the $\mathbf{F}_p$-dual space of D, is determined by assigning to each e_n an arbitrary element of $\mathbf{F}_p$ (why?), so $D^\vee$ is isomorphic as an $\mathbf{F}_p$-vector space to the direct product $\prod_{n\geq 1}\mathbf{F}_p$. Thus $D^\vee \cong G$ as $\mathbf{F}_p$-vector spaces.

Note D is naturally a subset of G, and D is a dense subset of G when G has the product topology. Each continuous homomorphism $G \to \mathbf{F}_p$ is determined by its values on a dense subset of G, such as D (so "D" denotes both direct sum and dense). Thus each continuous homomorphism $G \to \mathbf{F}_p$ is determined by its restriction to a homomorphism $D \to \mathbf{F}_p$, which is an element of $D^\vee$. Hence the set of continuous homomorphisms $G \to \mathbf{F}_p$ embeds into $D^\vee$ as a set. Thus the cardinality of the set of continuous homomorphisms $G \to \mathbf{F}_p$ is at most the cardinality of $D^\vee$. Since $D^\vee \cong G$ as $\mathbf{F}_p$-vector spaces, the cardinality of the set of continuous homomorphisms $G \to \mathbf{F}_p$ is at most the cardinality of G.

Let W be an infinite-dimensional $\mathbf{F}_p$-vector space, By Step 2 in the proof of Theorem 1 with $K = \mathbf{F}_p$,

$$\mathrm{card}(W) = \max(p, \mathrm{card}(\dim_{\mathbf{F}_p}(W))) = \mathrm{card}(\dim_{\mathbf{F}_p}(W)).$$

Since W is infinite dimensional, so is $W^\vee$. Thus with $W^\vee$ in place of W,

$$\mathrm{card}(W^\vee) = \mathrm{card}(\dim_{\mathbf{F}_p}(W^\vee)).$$

By Theorem 1, $\mathrm{card}(\dim_{\mathbf{F}_p}(W)) < \mathrm{card}(\dim_{\mathbf{F}_p}(W^\vee))$, so

$$\mathrm{card}(W) < \mathrm{card}(W^\vee).$$

Taking $W = G$, we have

$$\mathrm{card}(G) < \mathrm{card}(G^\vee).$$

Earlier we showed the set of continuous homomorphisms $G \to \mathbf{F}_p$ is at most the cardinality of G. Therefore the cardinality of the set of *continuous* homomorphisms $G \to \mathbf{F}_p$ is less than the cardinality of the set of *all* homomorphisms $G \to \mathbf{F}_p$ (which is $G^\vee$), so discontinuous homomorphisms $G \to \mathbf{F}_p$ exist. Thus non-open index-p subgroups of G (with the product topology) exist. □

References

[1] Alcides Buss, answer to MathOverflow question 13322, Slick proof?: A vector space has the same dimension as its dual if and only if it is finite dimensional, http://mathoverflow.net/questions/13322.

[2] Lukas Heger, answer to Math Stackexchange question 4963865, Non-open subgroups of finite index of $(\widehat{Z})^\times$, https://math.stackexchange.com/questions/4963865.