COMPLEXIFICATION

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1. Introduction

We want to describe a procedure for enlarging real vector spaces to complex vector spaces in a natural way. For instance, the natural complex analogues of $\mathbf{R}^n$, $\mathrm{M}_n(\mathbf{R})$, and $\mathbf{R}[X]$ are $\mathbf{C}^n$, $\mathrm{M}_n(\mathbf{C})$ and $\mathbf{C}[X]$.

Why do we want to complexify real vector spaces? One reason is related to solving equations. If we want to prove theorems about real solutions to a system of real linear equations or a system of real linear differential equations, it can be convenient as a first step to examine the complex solution space. Then we would try to use our knowledge of the complex solution space (for instance, its dimension) to get information about the real solution space. In the other direction, we may want to know if a subspace of $\mathbf{C}^n$ which is given to us as the solution set to a system of complex linear equations is also the solution set to a system of real linear equations. We will find a nice way to describe such subspaces once we understand the *different* ways that a complex vector space can occur as the complexification of a real subspace.

We will give two descriptions of the complexification process, first in terms of a two-fold direct sum (Section 2) and then in terms of tensor products (Section 3). The tensor product viewpoint is the more far-reaching one, but seeing how the direct sum method of complexification expresses everything may help convince the reader that there is nothing unexpected about this use tensor products. Moreover, idiosyncratic aspects of the direct sum construction will turn out to be completely natural features of the tensor product construction. After comparing the two constructions, we will see (Section 4) how to use a special structure on a complex vector space, called a conjugation, to describe the real subspaces which have the given complex vector space as its complexification.

References on this topic are [1, pp. 79–81], [2, §77], and [3, pp. 262–263]. In [1] and [3] tensor products are used, while [2] uses direct sums.

2. Complexifying with Direct Sums

Let W be a real vector space. To create from W a naturally associated complex vector space, we need to give a meaning to $(a+bi)w$, where $a+bi \in \mathbf{C}$ and $w \in W$. Whatever it might mean, we'd like to have

$$(a+bi)w = aw + biw = aw + ibw,$$

and since there is no given meaning to ibw (that is, multiplying elements of W by i has no definition since W is a real vector space), $aw + ibw$ should be thought of as a formal sum of aw and bw with the i factor keeping them apart. A legitimate way to make sense of this is to treat $aw + ibw$ as the ordered pair (aw, bw), and that leads to the following definition.

Definition 2.1. The *complexification* of a real vector space W is defined to be $W_{\mathbf{C}} = W \oplus W$, with multiplication law $(a+bi)(w_1, w_2) = (aw_1 - bw_2, bw_1 + aw_2)$, where a and b are real.

This rule of multiplication is reasonable if you think about a pair (w_1, w_2) in $W \oplus W$ as a formal sum $w_1 + iw_2$:

$$(a+bi)(w_1 + iw_2) = aw_1 + aiw_2 + biw_1 - bw_2 = (aw_1 - bw_2) + i(bw_1 + aw_2).$$

In particular,

$$i(w_1, w_2) = (-w_2, w_1). \tag{2.1}$$

It is left to the reader to check that $W_{\mathbf{C}}$ is a complex vector space by the multiplication rule in Definition 2.1 (*e.g.*, $z(z'(w_1, w_2)) = zz'(w_1, w_2)$). Since

$$i(w, 0) = (0, w),$$

we have

$$(w_1, w_2) = (w_1, 0) + (0, w_2) = (w_1, 0) + i(w_2, 0). \tag{2.2}$$

Thus elements of $W_{\mathbf{C}}$ formally look like $w_1 + iw_2$, except iw_2 has no meaning while $i(w_2, 0)$ does: it is $(0, w_2)$.

The real subspaces $W \oplus \{0\}$ and $\{0\} \oplus W$ of $W_{\mathbf{C}}$ both behave like W, since addition is componentwise and $a(w, 0) = (aw, 0)$ and $a(0, w) = (0, aw)$ when a is real. The $\mathbf{R}$-linear function $w \mapsto (w, 0)$ will be called the *standard embedding* of W into $W_{\mathbf{C}}$. So we treat $W \oplus \{0\}$ as the "official" copy of W inside $W_{\mathbf{C}}$ and with this identification made we can regard $W_{\mathbf{C}}$ as $W + iW$ using (2.2).

Example 2.2. Taking $W = \mathbf{R}$, its complexification $\mathbf{R}_{\mathbf{C}}$ is the set of ordered pairs of real numbers (x, y) with $(a+bi)(x, y) = (ax - by, bx + ay)$. Since $(a+bi)(x+yi) = (ax - by) + (bx + ay)i$, $\mathbf{R}_{\mathbf{C}}$ is isomorphic to $\mathbf{C}$ as $\mathbf{C}$-vector spaces by $(x, y) \mapsto x + yi$.

Example 2.3. The complexifications of $\mathbf{R}^n$, $\mathrm{M}_n(\mathbf{R})$, and $\mathbf{R}[X]$ are isomorphic to $\mathbf{C}^n$, $\mathrm{M}_n(\mathbf{C})$, and $\mathbf{C}[X]$, by sending an ordered pair (w_1, w_2) in $\mathbf{R}^n \oplus \mathbf{R}^n$, $\mathrm{M}_n(\mathbf{R}) \oplus \mathrm{M}_n(\mathbf{R})$, or $\mathbf{R}[X] \oplus \mathbf{R}[X]$ to $w_1 + iw_2$ in $\mathbf{C}^n$, $\mathrm{M}_n(\mathbf{C})$, or $\mathbf{C}[X]$.

For instance, we can identify the complexification $(\mathbf{R}^2)_{\mathbf{C}}$ with $\mathbf{C}^2$ (as complex vector spaces) by $(w_1, w_2) \mapsto w_1 + iw_2 \in \mathbf{R}^2 + i\mathbf{R}^2 = \mathbf{C}^2$, and this sends the basis vectors $\binom{1}{0}$ and $\binom{0}{1}$ of $\mathbf{R}^2 \subset (\mathbf{R}^2)_{\mathbf{C}}$ to $\binom{1}{0}$ and $\binom{0}{1}$ in $\mathbf{C}^2$, which are the standard basis vectors of $\mathbf{C}^2$ as a complex vector space. More generally, the identifications of $(\mathbf{R}^n)_{\mathbf{C}}, \mathrm{M}_n(\mathbf{R})_{\mathbf{C}}$, and $\mathbf{R}[X]_{\mathbf{C}}$ with $\mathbf{C}^n$, $\mathrm{M}_n(\mathbf{C})$, and $\mathbf{C}[X]$ turn every real basis of $\mathbf{R}^n$, $\mathrm{M}_n(\mathbf{R})$, and $\mathbf{R}[X]$ (viewed inside their complexifications by the standard embedding) into a complex basis of $\mathbf{C}^n$, $\mathrm{M}_n(\mathbf{C})$, and $\mathbf{C}[X]$.

Theorem 2.4. *If $W = 0$ then $W_{\mathbf{C}} = 0$. If $W \neq 0$ and $\{e_j\}$ is an $\mathbf{R}$-basis of W then $\{(e_j, 0)\}$ is a $\mathbf{C}$-basis of $W_{\mathbf{C}}$. In particular, $\dim_{\mathbf{C}}(W_{\mathbf{C}}) = \dim_{\mathbf{R}}(W)$ for all W.*

Proof. That W being zero implies $W_{\mathbf{C}}$ is zero is easy. Now take $W \neq 0$ with basis $\{e_j\}$.

For $(w_1, w_2) \in W_{\mathbf{C}}$, writing w_1 and w_2 as $\mathbf{R}$-linear combinations of the e_j's shows every element of $W_{\mathbf{C}}$ is an $\mathbf{R}$-linear combination of the $(e_j, 0)$'s and $(0, e_j)$'s. Since $(0, e_j) = i(e_j, 0)$, using $\mathbf{C}$-linear combinations we can write every element of $W_{\mathbf{C}}$ in terms of the vectors $(e_j, 0)$. Therefore $\{(e_j, 0)\}$ is a $\mathbf{C}$-linear spanning set of $W_{\mathbf{C}}$. To show it is linearly independent over $\mathbf{C}$ (and thus is a basis), suppose we can write $(0,0)$ as a (finite!) $\mathbf{C}$-linear combination of the vectors $(e_j, 0)$, say

$$(a_1 + ib_1)(e_1, 0) + \cdots + (a_m + ib_m)(e_m, 0) = (0,0)$$

for some real a_j and b_j. This is the same as

$$(a_1e_1 + \cdots + a_me_m, b_1e_1 + \cdots + b_me_m) = (0,0).$$

Therefore $\sum a_je_j = 0$ and $\sum b_je_j = 0$ in W. From linear independence of the e_j's over $\mathbf{R}$, all the coefficients a_j and b_j are 0, so $a_j + ib_j = 0$ for all j. $\square$

Example 2.5. Treating $\mathbf{C}$ as a real vector space, its complexification is *not* $\mathbf{C}$. Indeed $\mathbf{C}_{\mathbf{C}}$ has to have complex dimension 2 since $\mathbf{C}$ has real dimension 2 (Theorem 2.4). Explicitly, $\mathbf{C}_{\mathbf{C}} = \mathbf{C} \oplus \mathbf{C}$ with $(a + bi)(z_1, z_2) = (az_1 - bz_2, bz_1 + az_2)$. More generally, if W is a complex vector space the complexification $W_{\mathbf{C}}$ does not know about the original complex scaling on W; the construction of $W_{\mathbf{C}}$ only uses the real vector space structure of W and all information in advance about being able to multiply by i on W is gone.

A real $m \times n$ matrix, as an $\mathbf{R}$-linear transformation $\mathbf{R}^n \to \mathbf{R}^m$, can be viewed in a natural way as a function $\mathbf{C}^n \to \mathbf{C}^m$ and it becomes a $\mathbf{C}$-linear transformation. The next two theorems show how this process looks from the viewpoint of complexifications.

Theorem 2.6. *Every $\mathbf{R}$-linear transformation $\varphi\colon W \to W'$ of real vector spaces extends in a unique way to a $\mathbf{C}$-linear transformation of the complexifications: there is a unique $\mathbf{C}$-linear map $\varphi_{\mathbf{C}}\colon W_{\mathbf{C}} \to W'_{\mathbf{C}}$ making the diagram*

$$\begin{tikzcd} W \arrow[r, "\varphi"] \arrow[d] & W' \arrow[d] \\ W_{\mathbf{C}} \arrow[r, "\varphi_{\mathbf{C}}"'] & W'_{\mathbf{C}} \end{tikzcd}$$

commute, where the vertical maps are the standard embeddings of real vector spaces into their complexifications.

Proof. If such a $\mathbf{C}$-linear map $\varphi_{\mathbf{C}}$ exists, then the commutativity of the diagram says $\varphi_{\mathbf{C}}(w, 0) = (\varphi(w), 0)$ for all $w \in W$. Therefore when (w_1, w_2) is an element of $W_{\mathbf{C}}$,

$$\begin{aligned} \varphi_{\mathbf{C}}(w_1, w_2) &= \varphi_{\mathbf{C}}(w_1, 0) + \varphi_{\mathbf{C}}(0, w_2) \\ &= \varphi_{\mathbf{C}}(w_1, 0) + \varphi_{\mathbf{C}}(i(w_2, 0)) \\ &= \varphi_{\mathbf{C}}(w_1, 0) + i\varphi_{\mathbf{C}}(w_2, 0) \\ &= (\varphi(w_1), 0) + i(\varphi(w_2), 0) \\ &= (\varphi(w_1), 0) + (0, \varphi(w_2)) \\ &= (\varphi(w_1), \varphi(w_2)). \end{aligned}$$

This tells us what $\varphi_{\mathbf{C}}$ must be (notice the step where we used $\mathbf{C}$-linearity). So now we just turn around and *define* $\varphi_{\mathbf{C}}\colon W_{\mathbf{C}} \to W_{\mathbf{C}}$ by

$$\varphi_{\mathbf{C}}(w_1, w_2) := (\varphi(w_1), \varphi(w_2)).$$

We need to check $\varphi_{\mathbf{C}}$ is $\mathbf{C}$-linear. Since $\varphi_{\mathbf{C}}$ is clearly $\mathbf{R}$-linear, the only thing to check is that $\varphi_{\mathbf{C}}$ commutes with multiplication by i. This is left to the reader, using (2.1). $\square$

We call $\varphi_{\mathbf{C}}$ the *complexification* of φ.

Theorem 2.7. *Let W and W' be nonzero finite-dimensional real vector spaces. For an $\mathbf{R}$-linear transformation $\varphi\colon W \to W'$ and $\mathbf{R}$-bases $\{e_i\}$ and $\{e'_j\}$ of W and W', the matrix for φ with respect to these bases is the matrix for $\varphi_{\mathbf{C}}\colon W_{\mathbf{C}} \to W'_{\mathbf{C}}$ with respect to the $\mathbf{C}$-bases $\{(e_i, 0)\}$ and $\{(e'_j, 0)\}$ of the complexifications.*

Proof. The columns of the matrix for φ with respect to the bases $\{e_i\}$ and $\{e'_j\}$ are the coefficients which express the values of φ on each e_k as an $\mathbf{R}$-linear combination of the e'_j's. Similarly, the columns of the matrix for $\varphi_{\mathbf{C}}$ with respect to the bases $\{(e_i, 0)\}$ and $\{(e'_j, 0)\}$ are the coefficients which express the values of $\varphi_{\mathbf{C}}$ on each $(e_i, 0)$ as a $\mathbf{C}$-linear combination of the $(e'_j, 0)$'s. Since $\varphi_{\mathbf{C}}(e_i, 0) = (\varphi(e_i), 0)$, the linear combination of $\varphi(e_i)$ in terms of the e'_j's will give the same linear combination of $\varphi_{\mathbf{C}}(e_i, 0)$ in terms of the $(e'_j, 0)$'s. So the matrices of φ and $\varphi_{\mathbf{C}}$ with respect to these bases are equal. □

Example 2.8. Let $A \in \mathrm{M}_2(\mathbf{R})$ act on $\mathbf{R}^2$. The definition of $A_{\mathbf{C}}$ on $(\mathbf{R}^2)_{\mathbf{C}}$ is

$$A_{\mathbf{C}}(w_1, w_2) = (Aw_1, Aw_2),$$

and the isomorphism $f\colon (\mathbf{R}^2)_{\mathbf{C}} \to \mathbf{C}^2$ by $f(w_1, w_2) = w_1 + iw_2$ identifies $A_{\mathbf{C}}$ with the function

$$w_1 + iw_2 \mapsto Aw_1 + iAw_2$$

on $\mathbf{C}^2$. Since A as a matrix acting on complex vectors satisfies $Aw_1 + iAw_2 = A(w_1 + iw_2)$, the diagram

$$\begin{tikzcd}
(\mathbf{R}^2)_{\mathbf{C}} \arrow[r, "A_{\mathbf{C}}"] \arrow[d, "f"'] & (\mathbf{R}^2)_{\mathbf{C}} \arrow[d, "f"] \\
\mathbf{C}^2 \arrow[r, "A"] & \mathbf{C}^2
\end{tikzcd}$$

commutes. This is the sense in which the complexification $A_{\mathbf{C}}$ is just the matrix A acting on $\mathbf{C}^2$.

It is straightforward to check from the definitions that if $\varphi\colon W \to W'$ and $\psi\colon W' \to W''$ are $\mathbf{R}$-linear transformations then $(\psi \circ \varphi)_{\mathbf{C}} = \psi_{\mathbf{C}} \circ \varphi_{\mathbf{C}}$, so complexification of linear maps commutes with composition, and easily $(\mathrm{id}_W)_{\mathbf{C}} = \mathrm{id}_{(W_{\mathbf{C}})}$.

If U is a real subspace of W, then $U_{\mathbf{C}}$ is a complex subspace of $W_{\mathbf{C}}$ (check). In particular, associated to an $\mathbf{R}$-linear map $\varphi\colon W \to W'$ are the real subspaces $\ker \varphi \subset W$ and $\mathrm{im}\, \varphi \subset W'$. We also have the complexified $\mathbf{C}$-linear map $\varphi_{\mathbf{C}}\colon W_{\mathbf{C}} \to W'_{\mathbf{C}}$ and its kernel and image, which are $\mathbf{C}$-subspaces of $W_{\mathbf{C}}$ and $W'_{\mathbf{C}}$. The construction of kernels and images behaves nicely with respect to complexification:

Theorem 2.9. *If $\varphi\colon W \to W'$ is $\mathbf{R}$-linear, its complexification $\varphi_{\mathbf{C}}\colon W_{\mathbf{C}} \to W'_{\mathbf{C}}$ has kernel and image*

$$\ker(\varphi_{\mathbf{C}}) = (\ker \varphi)_{\mathbf{C}}, \quad \mathrm{im}(\varphi_{\mathbf{C}}) = (\mathrm{im}\, \varphi)_{\mathbf{C}}.$$

Proof. Since $\varphi_{\mathbf{C}}(w_1, w_2) = (\varphi(w_1), \varphi(w_2))$, the condition $(w_1, w_2) \in \ker \varphi_{\mathbf{C}}$ is equivalent to w_1 and w_2 lying in $\ker \varphi$, so

$$\ker(\varphi_{\mathbf{C}}) = (\ker \varphi) \oplus (\ker \varphi) \subset W \oplus W = W_{\mathbf{C}}.$$

Thus $\ker(\varphi_{\mathbf{C}}) = (\ker \varphi)_{\mathbf{C}}$ as a subset of $W_{\mathbf{C}}$.

The image of $\varphi_{\mathbf{C}}$ is $\varphi(W) \oplus \varphi(W)$, which is the complexification of $\varphi(W) = \mathrm{im}\, \varphi$. □

Corollary 2.10. *If $\varphi\colon W \to W'$ is $\mathbf{R}$-linear, then $\varphi_{\mathbf{C}}$ is injective if and only if φ is injective, and $\varphi_{\mathbf{C}}$ is surjective if and only if φ is surjective.*

Proof. Injectivity of $\varphi_{\mathbf{C}}$ is equivalent to $\ker(\varphi_{\mathbf{C}}) = (\ker \varphi)_{\mathbf{C}}$ being 0, and a real vector space has complexification 0 if and only if it is 0. If φ is surjective then $\mathrm{im}(\varphi_{\mathbf{C}}) = (\mathrm{im}\,\varphi)_{\mathbf{C}} = W'_{\mathbf{C}}$, so $\varphi_{\mathbf{C}}$ is surjective. If φ is not surjective then its image $\varphi(W)$ is a proper subspace of W', so $\mathrm{im}(\varphi_{\mathbf{C}}) = \varphi(W) \oplus \varphi(W)$ is a proper subspace of $W' \oplus W'$ and therefore $\varphi_{\mathbf{C}}$ is not surjective. $\square$

As with most important constructions, we can describe the complexification $W_{\mathbf{C}}$ of a real vector space W by a universal mapping property. We have a standard embedding $W \to W_{\mathbf{C}}$, which is an $\mathbf{R}$-linear transformation. Consider now all $\mathbf{R}$-linear transformations $W \xrightarrow{f} V$ of the particular real vector space W into *complex* vector spaces V. The standard embedding $W \to W_{\mathbf{C}}$ is just one example, but it is really the most basic one, as the next theorem shows.

Theorem 2.11. *For each $\mathbf{R}$-linear map $W \xrightarrow{f} V$ from W into a complex vector space V, there is a unique $\mathbf{C}$-linear map $W_{\mathbf{C}} \xrightarrow{\widetilde{f}} V$ making the diagram*

$$\begin{tikzcd} & W_{\mathbf{C}} \arrow[dd, dotted, "\widetilde{f}"] \\ W \arrow[ur] \arrow[dr, "f"'] & \\ & V \end{tikzcd}$$

commute, where the map $W \to W_{\mathbf{C}}$ is the standard embedding.

Proof. This is quite similar to the proof of Theorem 2.6, so we just sketch the idea. Assuming $\widetilde{f}$ exists, show it satisfies

$$\widetilde{f}(w_1, w_2) = (f(w_1), f(w_2))$$

by $\mathbf{C}$-linearity. Now define $\widetilde{f}$ by this formula and check it is $\mathbf{C}$-linear and makes the diagram commute. $\square$

To put this in the language of universal mapping properties, if we take as objects all $\mathbf{R}$-linear maps $W \xrightarrow{f} V$ from the fixed $\mathbf{R}$-vector space W to varying $\mathbf{C}$-vector spaces V, a morphism from $W \xrightarrow{f_1} V_1$ to $W \xrightarrow{f_2} V_2$ is a $\mathbf{C}$-linear (not just $\mathbf{R}$-linear!) map $V_1 \xrightarrow{f'} V_2$ such that the diagram

$$\begin{tikzcd} & V_1 \arrow[dd, "f'"] \\ W \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & V_2 \end{tikzcd}$$

commutes. Then Theorem 2.11 says $W \to W_{\mathbf{C}}$ is an initial object (it admits a unique morphism to all other objects), so it is determined up to a unique isomorphism by the mapping property in Theorem 2.11.

3. Complexifying with Tensor Products

While the definition of $W_{\mathbf{C}}$ does not depend on a choice of basis for W, it implicitly depends on a choice of the real basis $\{1, i\}$ of $\mathbf{C}$ instead of other bases. This can be seen in the way multiplication by $\mathbf{C}$ scales pairs $(w_1, w_2) \in W_{\mathbf{C}}$, which we want to think of as a formal version of $w_1 + iw_2$. Although the universal mapping property in Theorem 2.11 gives the complexification construction a basis-free meaning (within the framework of mapping real vector spaces to complex vector spaces), it is also possible to give a completely basis-free construction of the complexification using tensor products. The idea is that $W_{\mathbf{C}}$ behaves like $\mathbf{C} \otimes_{\mathbf{R}} W$ and the complexification $\varphi_{\mathbf{C}} \colon W_{\mathbf{C}} \to W'_{\mathbf{C}}$ of an $\mathbf{R}$-linear map $\varphi \colon W \to W'$ behaves like the $\mathbf{C}$-linear map $1 \otimes \varphi \colon \mathbf{C} \otimes_{\mathbf{R}} W \to \mathbf{C} \otimes_{\mathbf{R}} W'$. Here are some similarities between $W_{\mathbf{C}}$ and $\mathbf{C} \otimes_{\mathbf{R}} W$:

1. There are standard embeddings $W \to W_{\mathbf{C}}$ by $w \mapsto (w, 0)$ and $W \to \mathbf{C} \otimes_{\mathbf{R}} W$ by $w \mapsto 1 \otimes w$, and with these embeddings we have $W_{\mathbf{C}} = W + iW$ and $\mathbf{C} \otimes_{\mathbf{R}} W = W + iW$.
2. For a nonzero real vector space W, each $\mathbf{R}$-basis $\{e_j\}$ of W gives us a $\mathbf{C}$-basis $\{1 \otimes e_j\}$ of $\mathbf{C} \otimes_{\mathbf{R}} W$, so the $\mathbf{C}$-dimension of $\mathbf{C} \otimes_{\mathbf{R}} W$ equals the $\mathbf{R}$-dimension of W. This looks like Theorem 2.4.
3. The identifications of $(\mathbf{R}^n)_{\mathbf{C}}$, $\mathrm{M}_n(\mathbf{R})_{\mathbf{C}}$, and $\mathbf{R}[X]_{\mathbf{C}}$ with $\mathbf{C}^n$, $\mathrm{M}_n(\mathbf{C})$, and $\mathbf{C}[X]$ in Example 2.3 remind us of the effect of base extension to $\mathbf{C}$ of $\mathbf{R}^n$, $\mathrm{M}_n(\mathbf{R})$, and $\mathbf{R}[X]$.
4. In the finite-dimensional case, the matrices for φ and $1 \otimes \varphi$ are equal when using compatible bases, just as in Theorem 2.7.
5. Theorem 2.9 is similar to the formulas
$$\ker(1 \otimes \varphi) = \mathbf{C} \otimes_{\mathbf{R}} \ker \varphi, \quad \mathrm{im}(1 \otimes \varphi) = \mathbf{C} \otimes_{\mathbf{R}} (\mathrm{im}\, \varphi)$$
for kernels and images under base extension.

The constructions of both $W_{\mathbf{C}}$ and $\mathbf{C} \otimes_{\mathbf{R}} W$ from W should be two different ways of thinking about the same thing. Let's make this official.

Theorem 3.1. *For every real vector space W, there is a unique isomorphism $f_W \colon W_{\mathbf{C}} \to \mathbf{C} \otimes_{\mathbf{R}} W$ of $\mathbf{C}$-vector spaces which makes the diagram*

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ W_{\mathbf{C}} \arrow[rr, "f_W"] & & \mathbf{C} \otimes_{\mathbf{R}} W \end{tikzcd}$$

commute, where the two arrows out of W are its standard embeddings. Explicitly,

$$f_W(w_1, w_2) = 1 \otimes w_1 + i \otimes w_2. \tag{3.1}$$

Moreover, if $\varphi \colon W \to W'$ is an $\mathbf{R}$-linear map of real vector spaces, the diagram of $\mathbf{C}$-linear maps

$$\begin{tikzcd} W_{\mathbf{C}} \arrow[r, "\varphi_{\mathbf{C}}"] \arrow[d, "f_W"'] & W'_{\mathbf{C}} \arrow[d, "f_{W'}"] \\ \mathbf{C} \otimes_{\mathbf{R}} W \arrow[r, "1 \otimes \varphi"] & \mathbf{C} \otimes_{\mathbf{R}} W' \end{tikzcd} \tag{3.2}$$

commutes.

Proof. Assuming f_W exists, we must have

$$\begin{aligned} f_W(w_1, w_2) &= f_W((w_1, 0) + i(w_2, 0)) \\ &= f_W(w_1, 0) + i f_W(w_2, 0) \\ &= 1 \otimes w_1 + i(1 \otimes w_2) \\ &= 1 \otimes w_1 + i \otimes w_2. \end{aligned}$$

Now if we define f_W by this last expression, it is easy to see f_W is **R**-linear. To see f_W is in fact **C**-linear, we compute

$$f_W(i(w_1, w_2)) = f_W(-w_2, w_1) = 1 \otimes (-w_2) + i \otimes w_1 = -1 \otimes w_2 + i \otimes w_1$$

and

$$i f_W(w_1, w_2) = i(1 \otimes w_1 + i \otimes w_2) = i \otimes w_1 + (-1) \otimes w_2 = -1 \otimes w_2 + i \otimes w_1.$$

To show f_W is an isomorphism, we write down the inverse map $\mathbf{C} \otimes_{\mathbf{R}} W \to W_{\mathbf{C}}$. We'd like $1 \otimes w$ to correspond to $(w, 0)$, so we expect $z \otimes w = z(1 \otimes w)$ should go to $z(w, 0)$. With this in mind, we construct such a map by first letting $\mathbf{C} \times W \to W_{\mathbf{C}}$ by $(z, w) \mapsto z(w, 0)$. This is **R**-bilinear, so it induces an **R**-linear map $g_W \colon \mathbf{C} \otimes_{\mathbf{R}} W \to W_{\mathbf{C}}$ where $g_W(z \otimes w) = z(w, 0)$ on elementary tensors. To show g_W is **C**-linear, it suffices to check $g_W(zt) = z g_W(t)$ when t is an elementary tensor, say $t = z' \otimes w$:

$$g_W(z(z' \otimes w)) = g_W(zz' \otimes w) = zz'(w, 0), \quad z g_W(z' \otimes w) = z(z'(w, 0)) = zz'(w, 0).$$

Finally, to show f_W and g_W are inverses, first we have

$$g_W(f_W(w_1, w_2)) = g_W(1 \otimes w_1 + i \otimes w_2) = (w_1, 0) + i(w_2, 0) = (w_1, 0) + (0, w_2) = (w_1, w_2).$$

To show $f_W(g_W(t)) = t$ for all $t \in \mathbf{C} \otimes_{\mathbf{R}} W$, it suffices to verify this on elementary tensors, and this is left to the reader. The commutativity of (3.2) is also left to the reader. □

Since $W_{\mathbf{C}}$ and $\mathbf{C} \otimes_{\mathbf{R}} W$ are both the direct sum of subspaces W and iW (using the standard embedding of W into $W_{\mathbf{C}}$ and $\mathbf{C} \otimes_{\mathbf{R}} W$), this suggests saying a complex vector space V *is the complexification of a real subspace* W when $V = W + iW$ and $W \cap iW = \{0\}$. For example, in this sense $\mathbf{C}^n$ is the complexification of $\mathbf{R}^n$. The distinction from the preceding meaning of complexification is that now we are talking about how a pre-existing complex vector space can be a complexification of a real subspace of it, rather than starting with a real vector space and trying to create a complexification out of it.

Theorem 3.2. *Let V be a nonzero complex vector space and W be a real nonzero subspace. The following are equivalent:*

(1) *V is the complexification of W,*
(2) *every **R**-basis of W is a **C**-basis of V,*
(3) *some **R**-basis of W is a **C**-basis of V,*
(4) *the **C**-linear map $\mathbf{C} \otimes_{\mathbf{R}} W \to V$ given by $z \otimes w \mapsto zw$ on elementary tensors is an isomorphism of complex vector spaces,*

Proof. (1) ⇒ (2): Let $\{e_j\}$ be an **R**-basis of W. Since $V = W + iW$, V is spanned over **R** by $\{e_j, ie_j\}$, so V is spanned over **C** by $\{e_j\}$. To show linear independence of $\{e_j\}$ over **C**, suppose $\sum(a_j + ib_j)e_j = 0$. Then $\sum a_j e_j = i \sum(-b_j)e_j \in W \cap iW = \{0\}$, so $a_j = 0$ and $b_j = 0$ for all j.

(2) ⇒ (3): Obvious.

(3) $\Rightarrow$ (4): If some $\mathbf{R}$-basis $\{e_j\}$ of W is a $\mathbf{C}$-basis of V, then the $\mathbf{C}$-linear map $\mathbf{C}\otimes_{\mathbf{R}} W \to V$ given by $z \otimes w \mapsto zw$ sends $1 \otimes e_j$ to e_j so it identifies $\mathbf{C}$-bases of two complex vector spaces. Therefore this map is an isomorphism of complex vector spaces.

(4) $\Rightarrow$ (1): Since $\mathbf{C}\otimes_{\mathbf{R}} W = 1\otimes W + i(1\otimes W)$ and $(1\otimes W)\cap i(1\otimes W) = 0$, the isomorphism with V shows V is a complexification of W. $\square$

By Theorem 3.2, every complex vector space V is the complexification of some real subspace W: if $V \neq 0$, pick a $\mathbf{C}$-basis $\{e_j\}$ of V and take for W the *real* span of the e_j's in V. This is trivial if $V = 0$.

Example 3.3. Two real subspaces of $\mathrm{M}_2(\mathbf{C})$ having $\mathrm{M}_2(\mathbf{C})$ as their complexification are $\mathrm{M}_2(\mathbf{R})$ and $\{\left(\begin{smallmatrix} a & b+ci \\ b-ci & d \end{smallmatrix}\right) : a,b,c,d \in \mathbf{R}\}$: both are 4-dimensional real subspaces containing a $\mathbf{C}$-basis of $\mathrm{M}_2(\mathbf{C})$ (check!).

Note that saying V has some $\mathbf{R}$-basis of W as a $\mathbf{C}$-basis is *stronger* than saying V has a $\mathbf{C}$-spanning set from W, even though "a spanning set contains a basis": a spanning set for V in W means $\mathbf{C}$-coefficients, while a basis of W means $\mathbf{R}$-coefficients, so there is a mismatch. As an example, let $V = W = \mathbf{C}$. Then V has a $\mathbf{C}$-spanning set in W, *e.g.*, $\{1\}$, but no $\mathbf{R}$-basis of W is a $\mathbf{C}$-basis of V. In general, a spanning set for V from W need not be a basis of W. But see Corollary 4.12 for conditions where a spanning set plays an important role.

Since both $W_{\mathbf{C}}$ and $\mathbf{C}\otimes_{\mathbf{R}} W$ have the same properties relative to W insofar as complexifications are concerned, we can use the label "complexification" for either construction. In particular, when referring to the complexification of an $\mathbf{R}$-linear map $\varphi\colon W \to W'$, we can mean either $\varphi_{\mathbf{C}}\colon W_{\mathbf{C}} \to W'_{\mathbf{C}}$ or $1\otimes\varphi\colon \mathbf{C}\otimes_{\mathbf{R}} W \to \mathbf{C}\otimes_{\mathbf{R}} W'$.

4. Conjugations on Complex Vector Spaces

Real subspaces with a given complexification have a name:

Definition 4.1. If V is a complex vector space, an $\mathbf{R}$-*form* of V is a real subspace W of V having V as its complexification.

Concretely, an $\mathbf{R}$-form of V is the $\mathbf{R}$-span of a $\mathbf{C}$-basis of V.

Example 4.2. Two $\mathbf{R}$-forms of $\mathrm{M}_2(\mathbf{C})$ are $\mathrm{M}_2(\mathbf{R})$ and$\{\left(\begin{smallmatrix} a & b+ci \\ b-ci & d \end{smallmatrix}\right) : a,b,c,d \in \mathbf{R}\}$.

There is a way to keep track of all the $\mathbf{R}$-forms of a complex vector space by using a generalization of complex conjugation. When we base extend a real vector space W to the complex vector space $\mathbf{C}\otimes_{\mathbf{R}} W$, complex conjugation can be extended from $\mathbf{C}$ to the tensor product: let $\tau_W\colon \mathbf{C}\otimes_{\mathbf{R}} W \to \mathbf{C}\otimes_{\mathbf{R}} W$ by

$$\tau_W(z\otimes w) := \overline{z}\otimes w. \tag{4.1}$$

This is $\mathbf{R}$-linear, and we can recover W (or rather $1\otimes W$ inside $\mathbf{C}\otimes_{\mathbf{R}} W$) from τ_W by taking fixed points. If $w \in W$ then$\tau_W(1\otimes w) = 1\otimes w$. In the other direction, if $\tau_W(t) = t$ for some $t \in \mathbf{C}\otimes_{\mathbf{R}} W$, we want to show $t \in 1\otimes W$. Write $t = 1\otimes w_1 + i\otimes w_2$; every element of $\mathbf{C}\otimes_{\mathbf{R}} W$ has this form in a unique way. Then $\tau_W(t) = 1\otimes w_1 - i\otimes w_2$, so $\tau_W(t) = t$ if and only if $i\otimes w_2 = -i\otimes w_2$, which is the same as $2i\otimes w_2 = 0$, and that implies $w_2 = 0$, so $t = 1\otimes w_1 \in 1\otimes W$. That the real vector space W (really, its isomorphic standard copy $1\otimes W$ in $\mathbf{C}\otimes_{\mathbf{R}} W$) is the set of fixed points of τ_W generalizes the fact that $\mathbf{R}$ is the set of fixed points of complex conjugation on $\mathbf{C}$: whatever is fixed by something like complex conjugation should be thought of as a "real" object.

Remark 4.3. If we think of the complexification of W as $W_{\mathbf{C}} = W \oplus W$, it is natural to call the complex conjugate of $(w_1, w_2) = (w_1, 0) + i(w_2, 0)$ the vector $(w_1, 0) - i(w_2, 0) = (w_1, -w_2)$, so we set $\overline{(w_1, w_2)} = (w_1, -w_2)$. The reader can check the diagram

$$\begin{array}{ccc} W_{\mathbf{C}} & \xrightarrow{f_W} & \mathbf{C} \otimes_{\mathbf{R}} W \\ {\scriptstyle v \mapsto \overline{v}} \downarrow & & \downarrow {\scriptstyle \tau_W} \\ W_{\mathbf{C}} & \xrightarrow{f_W} & \mathbf{C} \otimes_{\mathbf{R}} W \end{array}$$

commutes, where f_W is the isomorphism in Theorem 3.1. It is easy to see on $W_{\mathbf{C}}$ that the fixed points of the conjugation are $\{(w, 0)\}$, which is the standard copy of W in $W_{\mathbf{C}}$.

Abstracting the function τ_W on $\mathbf{C} \otimes_{\mathbf{R}} W$ to a complex vector space V, we are led to the following coordinate-free concept.

Definition 4.4. A *conjugation* on a complex vector space V is a function $c\colon V \to V$ such that

- $c(v + v') = c(v) + c(v')$ for all v and v' in V,
- $c(zv) = \overline{z}c(v)$ for all $z \in \mathbf{C}$ and $v \in V$,
- $c(c(v)) = v$ for all $v \in V$.

A conjugation is $\mathbf{R}$-linear (take $z \in \mathbf{R}$ in the second property) but is not $\mathbf{C}$-linear: $c(iv) = -ic(v)$. The first and second properties together are called *conjugate-linearity*.

Example 4.5. On $\mathbf{C}^n$, the function $c(z_1, \ldots, z_n) = (\overline{z}_1, \ldots, \overline{z}_n)$, where every coordinate gets replaced by its complex conjugate, is a conjugation. In terms of the standard basis $e_1, \ldots, e_n$ of $\mathbf{C}^n$, this conjugation is given by $c(\sum z_j e_j) = \sum \overline{z}_j e_j$. Its fixed points are $\mathbf{R}^n$.

Example 4.6. On $\mathbf{C}[X]$, the function $c(\sum_{j=0}^d a_j X^j) = \sum_{j=0}^d \overline{a}_j X^j$ is a conjugation. Its fixed points are $\mathbf{R}[X]$.

Example 4.7. On $\mathrm{M}_2(\mathbf{C})$, two examples of conjugations are $c\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) = \left(\begin{smallmatrix} \overline{\alpha} & \overline{\beta} \\ \overline{\gamma} & \overline{\delta} \end{smallmatrix}\right)$ and $c\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) = \left(\begin{smallmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{smallmatrix}\right)$. The first one conjugates all the matrix entries, and the second one is the conjugate-transpose operation. The fixed points of these conjugations are $\mathrm{M}_2(\mathbf{R})$ and$\{\left(\begin{smallmatrix} a & b+ci \\ b-ci & d \end{smallmatrix}\right) : a, b, c, d \in \mathbf{R}\}$, which we met before (Example 3.3) as two $\mathbf{R}$-forms of $\mathrm{M}_2(\mathbf{C})$.

Example 4.8. For a real vector space W, the function τ_W in (4.1) is called the *standard conjugation* on $\mathbf{C} \otimes_{\mathbf{R}} W$.

Example 4.9. Under the usual $\mathbf{C}$-vector space isomorphisms $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{R}^n \cong \mathbf{C}^n$ and $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{R}[X] \cong \mathbf{C}[X]$, the standard conjugations on the tensor products are identified with the conjugations on $\mathbf{C}^n$ and $\mathbf{C}[X]$ from Examples 4.5 and 4.6. That is, the diagrams

$$\begin{array}{ccc} \mathbf{C} \otimes_{\mathbf{R}} \mathbf{R}^n & \longrightarrow & \mathbf{C}^n \\ \downarrow & & \downarrow \\ \mathbf{C} \otimes_{\mathbf{R}} \mathbf{R}^n & \longrightarrow & \mathbf{C}^n \end{array} \qquad \begin{array}{ccc} \mathbf{C} \otimes_{\mathbf{R}} \mathbf{R}[X] & \longrightarrow & \mathbf{C}[X] \\ \downarrow & & \downarrow \\ \mathbf{C} \otimes_{\mathbf{R}} \mathbf{R}[X] & \longrightarrow & \mathbf{C}[X] \end{array}$$

commute, where the horizontal maps are the usual isomorphisms, the left vertical arrows are the standard conjugations, and the right vertical arrows are the conjugations introduced on $\mathbf{C}^n$ and $\mathbf{C}[X]$ in Examples 4.5 and 4.6. .

Example 4.10. If $f\colon V_1 \to V_2$ is an isomorphism of complex vector spaces then a conjugation on V_1 determines a unique conjugation on V_2 such that f transforms the conjugation on V_1 into that on V_2. That is, if the left vertical arrow c_1 in the diagram

$$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[d, "c_1"'] & V_2 \arrow[d, dashed, "c_2"] \\ V_1 \arrow[r, "f"'] & V_2 \end{tikzcd}$$

is a conjugation on V_1 then there is exactly one conjugation c_2 on V_2 that makes the diagram commute: $f(c_1(v)) = c_2(f(v))$ for all $v \in V_1$. Indeed, for each $v' \in V_2$ there is $v \in V_1$ such that $v' = f(v)$. Then $c_2(v') = c_2(f(v)) = f(c_1(v)) = f(c_1(f^{-1}(v')))$, so we must have $c_2 = fc_1f^{-1}$. Check this formula does define a conjugation on V_2. In particular, if c is a conjugation on a complex vector space V and $A \in \mathrm{GL}(V)$ then A is an isomorphism of V with itself and $c'(v) := Ac(A^{-1}v)$ is another conjugation on V: the diagram

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "c"'] & V \arrow[d, "c'"] \\ V \arrow[r, "A"'] & V \end{tikzcd} \tag{4.2}$$

commutes and c' is the only conjugation for which that is true.

Example 4.7 reinforces the point that a complex vector space does not have a unique conjugation on it. Still, you may feel that the conjugation in Example 4.5 is the most natural choice on $\mathbf{C}^n$. And you may think $\mathbf{R}^n$ is the most natural $\mathbf{R}$-form of $\mathbf{C}^n$. These two feelings are related! Here is the link between them, in the most general case:

Theorem 4.11. *Let V be a complex vector space. There is a bijection between the following data on V:*

1. $\mathbf{R}$-*forms of V,*
2. *conjugations on V.*

Proof. First we will explain what the maps are in both directions. Then we check the maps are inverses.

If we start with an $\mathbf{R}$-form W of V, so there is an isomorphism $f\colon \mathbf{C}\otimes_{\mathbf{R}} W \to V$ by $f(z\otimes w) = zw$, we can use this isomorphism to transport the standard conjugation τ_W on $\mathbf{C}\otimes_{\mathbf{R}} W$ to a conjugation c_W on V (Example 4.10). That is, we can uniquely fill in the arrow on the right of the diagram

$$\begin{tikzcd} \mathbf{C}\otimes_{\mathbf{R}} W \arrow[r, "f"] \arrow[d, "\tau_W"'] & V \arrow[d, dashed, "c_W"] \\ \mathbf{C}\otimes_{\mathbf{R}} W \arrow[r, "f"'] & V \end{tikzcd}$$

to make it commute. Explicitly, $c_W(\sum_i z_iw_i) = \sum_i \overline{z}_iw_i$ where $z_i \in \mathbf{C}$ and $w_i \in W$. (Considering the many ways of writing an element of V in the form $\sum_i z_iw_i$ – the w_i's need not be a basis – the fact that this formula for c_W is well-defined depends crucially on $\mathbf{C}\otimes_{\mathbf{R}} W$ being isomorphic to V by $z\otimes w \mapsto zw$, which is what the two horizontal arrows are in the diagram.) So from an $\mathbf{R}$-form W of V we get a conjugation c_W on V.

If instead we start with a conjugation c on V, an $\mathbf{R}$-form of V is

$$V_c = \{v \in V : c(v) = v\}. \tag{4.3}$$

To check this is an $\mathbf{R}$-form of V, first note V_c is an $\mathbf{R}$-subspace of V since c is $\mathbf{R}$-linear. Then we define an $\mathbf{R}$-linear map $f\colon \mathbf{C} \otimes_{\mathbf{R}} V_c \to V$ by $f(z \otimes w) = zw$. We want to show this is a $\mathbf{C}$-linear isomorphism. It's $\mathbf{C}$-linear since

$$f(z'(z \otimes w)) = f(z'z \otimes w) = (z'z)w = z'(zw) = z'f(z \otimes w),$$

so $f(z't) = z'f(t)$ for all $t \in \mathbf{C} \otimes_{\mathbf{R}} V_c$ by additivity.

To check f is onto, we can write each $v \in V$ as $v = w_1 + iw_2$, where $w_1 = (v + c(v))/2$ and $w_2 = (v - c(v))/2i$ are both fixed by c, so each lies in V_c. (Compare with the formulas $(z + \overline{z})/2$ and $(z - \overline{z})/2i$ for the real and imaginary parts of a complex number z.) Then $v = w_1 + iw_2 = f(1 \otimes w_1 + i \otimes w_2)$.

To check f is one-to-one, suppose $f(t) = 0$ for some $t \in \mathbf{C} \otimes_{\mathbf{R}} V_c$. We can write $t = 1 \otimes w + i \otimes w'$ for some w and w' in V_c, so the condition $f(t) = 0$ says $w + iw' = 0$ in V. Applying c to this yields $w - iw' = 0$ because w and w' are fixed by c. Adding and subtracting the equations $w + iw' = 0$ and $w - iw' = 0$ shows $w = 0$ and $w' = 0$, so $t = 0$. Thus f is one-to-one, so V_c is an $\mathbf{R}$-form of V.

It remains to check that our correspondences between $\mathbf{R}$-forms of V and conjugations on V are inverses of one another.

$\mathbf{R}$-form to conjugation and back: If we start with an $\mathbf{R}$-form W of V, we get a conjugation c_W on V by $c_W(\sum_i z_i w_i) := \sum_i \overline{z}_i w_i$ for $z_i \in \mathbf{C}$ and $w_i \in W$. The subspace V_{c_W} is $\{v \in V : c_W(v) = v\}$, so we need to check this is W. Certainly $W \subset V_c$.

Since $\mathbf{C} \otimes_{\mathbf{R}} W \cong V$ by $z \otimes w \mapsto zw$, every $v \in V$ is $w_1 + iw_2$ for some w_1 and w_2 in W. Then $c_W(v) = c_W(w_1 + iw_2) = c_W(w_1) + c_W(iw_2) = w_1 - iw_2$, so $c_W(v) = v$ if and only if $iw_2 = -iw_2$, which is equivalent to $w_2 = 0$, which means $v = w_1 \in W$. Thus $V_{c_W} \subset W$, so $W = V_c$.

Conjugation to $\mathbf{R}$-form and back: If we start with a conjugation c on V, so $\mathbf{C} \otimes_{\mathbf{R}} V_c \cong V$ by $z \otimes w \mapsto zw$, then we have to check this isomorphism transports the standard conjugation τ_{V_c} on $\mathbf{C} \otimes_{\mathbf{R}} V_c$ to the original conjugation c on V (and not some other conjugation on V). A tensor $\sum_i z_i \otimes w_i$ in $\mathbf{C} \otimes_{\mathbf{R}} V_c$ is identified with $\sum_i z_i w_i$ in V, and $\tau_{V_c}(\sum_i z_i \otimes w_i) = \sum_i \overline{z}_i \otimes w_i$ goes over to $\sum_i \overline{z}_i w_i$ in V. Since the w_i's are in V_c, $\sum_i \overline{z}_i w_i = \sum_i \overline{z}_i c(w_i) = c(\sum_i z_i w_i)$, so the isomorphism from $\mathbf{C} \otimes_{\mathbf{R}} V_c$ to V does transport τ_{V_c} on $\mathbf{C} \otimes_{\mathbf{R}} V_c$ to c on V. □

Corollary 4.12. *Let V be a complex subspace of $\mathbf{C}^n$ and c be the usual conjugation on $\mathbf{C}^n$. The following conditions are equivalent:*

1. *V has a $\mathbf{C}$-spanning set in $\mathbf{R}^n$,*
2. *$c(V) = V$,*
3. *V has an $\mathbf{R}$-form in $\mathbf{R}^n$,*
4. *$\dim_{\mathbf{R}}(V \cap \mathbf{R}^n) = \dim_{\mathbf{C}}(V)$*

When this happens, the only $\mathbf{R}$-form of V in $\mathbf{R}^n$ is $V \cap \mathbf{R}^n$.

Proof. This is all clear when V is 0, so take $V \neq 0$.

(1) ⇔ (2): If V has a $\mathbf{C}$-spanning set $\{v_j\}$ in $\mathbf{R}^n$, $c(V) = c(\sum_{j=1}^d \mathbf{C}v_j) = \sum_{j=1}^d \mathbf{C}c(v_j) = \sum_{j=1}^d \mathbf{C}v_j = V$. Conversely, if $c(V) = V$ then c is a conjugation on V and the corresponding $\mathbf{R}$-form of V is $V_c \subset (\mathbf{C}^n)_c = \mathbf{R}^n$. An $\mathbf{R}$-basis of V_c is a $\mathbf{C}$-basis of V, so V has a $\mathbf{C}$-basis in $\mathbf{R}^n$.

(2) $\Leftrightarrow$ (3): In the proof that (2) implies (1) we showed (2) implies (3). In the other direction, if (3) holds then there is an $\mathbf{R}$-subspace $W \subset \mathbf{R}^n$ such that V is the $\mathbf{C}$-span of W. That implies $c(V) = V$, since V has a $\mathbf{C}$-spanning in $\mathbf{R}^n$.

(1) $\Leftrightarrow$ (4): If V has a $\mathbf{C}$-spanning set in $\mathbf{R}^n$ then V has a $\mathbf{C}$-basis in $\mathbf{R}^n$. Let $d = \dim_{\mathbf{C}}(V)$, and $v_1, \dots, v_d$ be a $\mathbf{C}$-basis of V in $\mathbf{R}^n$. Then every $v \in V$ has the form $v = \sum a_j v_j$ with $a_j \in \mathbf{C}$. If $v \in \mathbf{R}^n$ then $c(v) = v$, so $\overline{a}_j = a_j$ (from linear independence of the v_j's over $\mathbf{C}$). Therefore v is in the $\mathbf{R}$-span of the v_j's, so $V \cap \mathbf{R}^n = \sum \mathbf{R} v_j$ has $\mathbf{R}$-dimension d. Conversely, if $\dim_{\mathbf{R}}(\mathbf{R}^n \cap V) = \dim_{\mathbf{C}}(V)$, let $v_1, \dots, v_d$ be an $\mathbf{R}$-basis of $V \cap \mathbf{R}^n$. Vectors in $\mathbf{R}^n$ that are linearly independent over $\mathbf{R}$ are linearly independent over $\mathbf{C}$, so the $\mathbf{C}$-span of the v_j's has $\mathbf{C}$-dimension d. Since this $\mathbf{C}$-span is in V, whose $\mathbf{C}$-dimension is d, the $\mathbf{C}$-span of the v_j's is V.

It remains to show under these 4 conditions that $V \cap \mathbf{R}^n$ is the unique $\mathbf{R}$-form of V in $\mathbf{R}^n$. By the proof of (4) $\Rightarrow$ (1), $V \cap \mathbf{R}^n$ is an $\mathbf{R}$-form of V. (This also follows from (2) $\Rightarrow$ (1) since $V_c = V \cap (\mathbf{C}^n)_c = V \cap \mathbf{R}^n$.) Now suppose $W \subset \mathbf{R}^n$ is an $\mathbf{R}$-form of V. Let $w_1, \dots, w_d$ be an $\mathbf{R}$-basis of W. The conjugation on V corresponding to the $\mathbf{R}$-form W is given by $\sum a_j w_j \mapsto \sum \overline{a}_j w_j = c(\sum a_j w_j)$ since $w_j \in \mathbf{R}^n$. By Theorem 4.11, $W = V \cap \mathbf{R}^n$. $\square$

To make Corollary 4.12 concrete, it says in part that the solutions to a homogeneous system of complex linear equations is spanned by its real solutions if and only if the solution space is preserved by the standard conjugation on $\mathbf{C}^n$ (in which case there is a homogeneous system of real linear equations with the same solutions).

Corollary 4.12 says $V \cap \mathbf{R}^n$ is an $\mathbf{R}$-form of V only when its $\mathbf{R}$-dimension has the largest conceivable size (since always $\dim_{\mathbf{R}}(V \cap \mathbf{R}^n) \leq \dim_{\mathbf{C}}(V)$). It is generally *false* that $V \cap \mathbf{R}^n$ spans V over $\mathbf{C}$, since the dimension of the intersection could be too small.

Example 4.13. Let $V = \mathbf{C}\binom{i}{1}$ in $\mathbf{C}^2$. Then $V \cap \mathbf{R}^2 = \{\binom{0}{0}\}$. An $\mathbf{R}$-form of this V is $\mathbf{R}\binom{i}{1}$, but there is no $\mathbf{R}$-form of V in $\mathbf{R}^2$.

There is a version of Corollary 4.12 for arbitrary $\mathbf{C}$-vector spaces, not just subspaces of $\mathbf{C}^n$ or finite-dimensional spaces:

Corollary 4.14. *Let U be a complex vector space with a conjugation c on it. For a $\mathbf{C}$-subspace $V \subset U$, the following conditions are equivalent:*

1. *V has a spanning set in U_c,*
2. *$c(V) = V$,*
3. *V has an $\mathbf{R}$-form in U_c.*

When these hold, the only $\mathbf{R}$-form of V in U_c is $V \cap U_c$. If $\dim_{\mathbf{C}}(V) < \infty$, these conditions are the same as $\dim_{\mathbf{C}}(V \cap U_c) = \dim_{\mathbf{C}}(V)$.

The proof is left as an exercise.

As another application of Theorem 4.11, we will prove a converse to the end of Example 4.10: all conjugations on a complex vector space are conjugate to each other by some (not unique) automorphism of the vector space.

Corollary 4.15. *Let V be a complex vector space with conjugations c and c'. There is some $A \in \mathrm{GL}(V)$ such that the diagram*

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "c"'] & V \arrow[d, "c'"] \\ V \arrow[r, "A"'] & V \end{tikzcd} \tag{4.4}$$

commutes.

Proof. Let $W = V_c$ and $W' = V_{c'}$. These are both $\mathbf{R}$-forms of V by Theorem 4.11, so the diagrams

$$\begin{tikzcd} \mathbf{C} \otimes_{\mathbf{R}} W \arrow[r] \arrow[d, "\tau_W"'] & V \arrow[d, "c"] \\ \mathbf{C} \otimes_{\mathbf{R}} W \arrow[r] & V \end{tikzcd} \qquad \begin{tikzcd} \mathbf{C} \otimes_{\mathbf{R}} W' \arrow[r] \arrow[d, "\tau_{W'}"'] & V \arrow[d, "c'"] \\ \mathbf{C} \otimes_{\mathbf{R}} W' \arrow[r] & V \end{tikzcd}$$

both commute, where the horizontal isomorphisms are given by $z \otimes w \mapsto zw$ and the left vertical maps are the standard conjugations (Example 4.8).

The real dimensions of W and W' are equal (since, as $\mathbf{R}$-forms of V, their real dimensions both equal $\dim_{\mathbf{C}}(V)$[1]), so there is an $\mathbf{R}$-linear isomorphism $\varphi \colon W \to W'$. The base extension $1 \otimes \varphi \colon \mathbf{C} \otimes_{\mathbf{R}} W \to \mathbf{C} \otimes_{\mathbf{R}} W'$ is a $\mathbf{C}$-linear isomorphism that respects the standard conjugation on each of these tensor products. That is, the diagram

$$\begin{tikzcd} \mathbf{C} \otimes_{\mathbf{R}} W \arrow[r, "1\otimes\varphi"] \arrow[d, "\tau_W"'] & \mathbf{C} \otimes_{\mathbf{R}} W' \arrow[d, "\tau_{W'}"] \\ \mathbf{C} \otimes_{\mathbf{R}} W \arrow[r, "1\otimes\varphi"'] & \mathbf{C} \otimes_{\mathbf{R}} W' \end{tikzcd}$$

commutes. (It suffices to check commutativity on elementary tensors: running along the top and right, $\tau_{W'}((1 \otimes \varphi)(z \otimes w)) = \tau_{W'}(z \otimes \varphi(w)) = \overline{z} \otimes \varphi(w)$, and running along the left and bottom $(1 \otimes \varphi)(\tau_W(z \otimes w)) = (1 \otimes \varphi)(\overline{z} \otimes w) = \overline{z} \otimes \varphi(w)$.) Now combine the three diagrams in this proof to get

$$\begin{tikzcd} V \arrow[r] \arrow[d, "c"'] & \mathbf{C} \otimes_{\mathbf{R}} W \arrow[r, "1\otimes\varphi"] \arrow[d, "\tau_W"'] & \mathbf{C} \otimes_{\mathbf{R}} W' \arrow[r] \arrow[d, "\tau_{W'}"'] & V \arrow[d, "c'"] \\ V \arrow[r] & \mathbf{C} \otimes_{\mathbf{R}} W \arrow[r, "1\otimes\varphi"'] & \mathbf{C} \otimes_{\mathbf{R}} W' \arrow[r] & V. \end{tikzcd}$$

The maps along the top and bottom are $\mathbf{C}$-linear isomorphisms. Call the (common) composite map along the top and bottom A, so $A \in \mathrm{GL}(V)$, and remove the middle vertical maps to be left with a commutative diagram of the form (4.4). $\square$

Conjugations can be used to describe how the complexifications of $\mathbf{R}$-linear maps $W_1 \to W_2$ sit inside the $\mathbf{C}$-linear maps $\mathbf{C} \otimes_{\mathbf{R}} W_1 \to \mathbf{C} \otimes_{\mathbf{R}} W_2$:

[1] This could be an infinite cardinal number, but what we do is still correct in that case.

Theorem 4.16. *A* $\mathbf{C}$*-linear map* $\Phi\colon \mathbf{C}\otimes_{\mathbf{R}} W_1 \to \mathbf{C}\otimes_{\mathbf{R}} W_2$ *has the form* $1\otimes\varphi$ *for some* $\mathbf{R}$*-linear map* $\varphi\colon W_1\to W_2$ *if and only if*

$$\Phi(\tau_{W_1}(t)) = \tau_{W_2}(\Phi(t)) \tag{4.5}$$

for all $t\in \mathbf{C}\otimes_{\mathbf{R}} W_1$*. Here* τ_{W_1} *and* τ_{W_2} *are the standard conjugations on* $\mathbf{C}\otimes_{\mathbf{R}} W_1$ *and* $\mathbf{C}\otimes_{\mathbf{R}} W_2$*.*

Proof. Suppose $\Phi = 1\otimes\varphi$ for some φ. To check $\Phi(\tau_{W_1}(t)) = \tau_{W_2}(\Phi(t))$ for all t, we only have to check it on elementary tensors $t = z\otimes w_1$: $\Phi(\tau_{W_1}(z\otimes w_1)) = (1\otimes\varphi)(\overline{z}\otimes w_1) = \overline{z}\otimes\varphi(w_1)$ and $\tau_{W_2}(\Phi(z\otimes w_1)) = \tau_{W_2}(z\otimes\varphi(w_1)) = \overline{z}\otimes\varphi(w_1)$, so (4.5) holds.

Conversely, if Φ satisfies (4.5) then at $t = 1\otimes w_1$ we get $\Phi(1\otimes w_1) = \tau_{W_2}(\Phi(1\otimes w_1))$. Thus $\Phi(1\otimes w_1)$ is fixed by the standard conjugation on $\mathbf{C}\otimes_{\mathbf{R}} W_2$, so $\Phi(1\otimes w_1)\in 1\otimes W_2$. Every element of $1\otimes W_1$ has the form $1\otimes w_2$ for exactly one w_2, so we can *define* $\varphi(w_1)\in W_2$ by

$$\Phi(1\otimes w_1) = 1\otimes(\varphi(w_1)).$$

This serves to define a function $\varphi\colon W_1\to W_2$. Since $\Phi(w_1+w_1') = \Phi(w_1)+\Phi(w_1')$,

$$1\otimes(\varphi(w_1+w_1')) = 1\otimes(\varphi(w_1)) + 1\otimes(\varphi(w_1')) = 1\otimes(\varphi(w_1)+\varphi(w_1')),$$

so $\varphi(w_1+w_1') = \varphi(w_1)+\varphi(w_1')$. In a similar way one can show φ commutes with scaling by real numbers, so φ is $\mathbf{R}$-linear.

Now we want to show $\Phi = 1\otimes\varphi$ as functions on $\mathbf{C}\otimes_{\mathbf{R}} W_1$. Both are $\mathbf{C}$-linear, so it suffices to compare them on elementary tensors of the form $1\otimes w$, where

$$\Phi(1\otimes w_1) = 1\otimes(\varphi(w_1)) = (1\otimes\varphi)(1\otimes w_1).$$

□

Theorem 4.16 is related to Theorem 4.11. Define $c\colon \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}\otimes_{\mathbf{R}} W_1, \mathbf{C}\otimes_{\mathbf{R}} W_2)\to \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}\otimes_{\mathbf{R}} W_1, \mathbf{C}\otimes_{\mathbf{R}} W_2)$ by

$$c(\Phi) := \tau_{W_2}\circ\Phi\circ\tau_{W_1}.$$

Check c is a conjugation on the complex vector space $\mathrm{Hom}_{\mathbf{C}}(\mathbf{C}\otimes_{\mathbf{R}} W_1, \mathbf{C}\otimes_{\mathbf{R}} W_2)$. Theorem 4.16 says c has a naturally associated $\mathbf{R}$-form: $\{\Phi : c(\Phi) = \Phi\}$. Saying $c(\Phi) = \Phi$ is the same as $\Phi\circ\tau_{W_1} = \tau_{W_2}\circ\Phi$ (since $\tau_{W_2}^{-1} = \tau_{W_1}$), which is (4.5). So Theorem 4.16 says the $\mathbf{R}$-form of $\mathrm{Hom}_{\mathbf{C}}(\mathbf{C}\otimes_{\mathbf{R}} W_1, \mathbf{C}\otimes_{\mathbf{R}} W_2)$ corresponding to the conjugation c is $\{1\otimes\varphi : \varphi\in \mathrm{Hom}_{\mathbf{R}}(W_1,W_2)\}$, which is $\mathrm{Hom}_{\mathbf{R}}(W_1,W_2)$ under the usual identifications.

Here is a more general version of Theorem 4.16, starting from complex vector spaces equipped with conjugations.

Theorem 4.17. *Let* V_1 *and* V_2 *be complex vector spaces equipped with conjugations* c_1 *and* c_2*. A* $\mathbf{C}$*-linear map* $\Phi\colon V_1\to V_2$ *has the form* $1\otimes\varphi$ *for some* $\mathbf{R}$*-linear map* $\varphi\colon (V_1)_{c_1}\to (V_2)_{c_2}$ *if and only if*

$$\Phi(c_1(t)) = c_2(\Phi(t))$$

for all $t\in V_1$

The proof is left as an exercise.

References

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