THE SYLOW THEOREMS

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1. Introduction

The converse of Lagrange's theorem is false: if G is a finite group and $d \mid |G|$, then there may not be a subgroup of G with order d. The simplest example of this is the group A_4, of order 12, which has no subgroup of order 6. The Norwegian mathematician Peter Ludwig Sylow [4] discovered that a converse result *is* true when d is a prime power: if p is a prime number and $p^k \mid |G|$ then G must contain a subgroup of order p^k. Sylow also discovered important relations among the subgroups whose order is the *largest* power of p dividing $|G|$, such as the fact that all subgroups of that order are conjugate to each other.

For example, a group of order $100 = 2^2 \cdot 5^2$ must contain subgroups of order 1, 2, 4, 5, and 25, the subgroups of order 4 are conjugate to each other, and the subgroups of order 25 are conjugate to each other. It is not necessarily the case that the subgroups of order 2 are conjugate or that the subgroups of order 5 are conjugate.

Definition 1.1. Let G be a finite group and p be a prime. A subgroup of G whose order is the highest power of p dividing $|G|$ is called a *p-Sylow subgroup*[1] of G. A p-Sylow subgroup for some p is called a *Sylow subgroup*.

In a group of order 100, a 2-Sylow subgroup has order 4, a 5-Sylow subgroup has order 25, and a p-Sylow subgroup is trivial if $p \neq 2$ or 5.

In a group of order 12, a 2-Sylow subgroup has order 4, a 3-Sylow subgroup has order 3, and a p-Sylow subgroup is trivial if $p > 3$. Let's look at a few examples of Sylow subgroups in groups of order 12.

Example 1.2. In $\mathbf{Z}/(12)$, the only 2-Sylow subgroup is $\{0, 3, 6, 9\} = \langle 3 \rangle$ and the only 3-Sylow subgroup is $\{0, 4, 8\} = \langle 4 \rangle$.

Example 1.3. In A_4 there is one subgroup of order 4, so the only 2-Sylow subgroup is

$$\{(1),\ (12)(34),\ (13)(24),\ (14)(23)\} = \langle (12)(34), (14)(23) \rangle.$$

There are four 3-Sylow subgroups:

$$\{(1),\ (123),\ (132)\} = \langle (123) \rangle, \quad \{(1),\ (124),\ (142)\} = \langle (124) \rangle,$$

$$\{(1),\ (134),\ (143)\} = \langle (134) \rangle, \quad \{(1),\ (234),\ (243)\} = \langle (234) \rangle.$$

Example 1.4. In D_6 there are three 2-Sylow subgroups:

$$\{1,\ r^3,\ s,\ r^3s\} = \langle r^3, s \rangle, \ \ \{1,\ r^3,\ rs,\ r^4s\} = \langle r^3, rs \rangle, \ \ \{1,\ r^3,\ r^2s,\ r^5s\} = \langle r^3, r^2s \rangle.$$

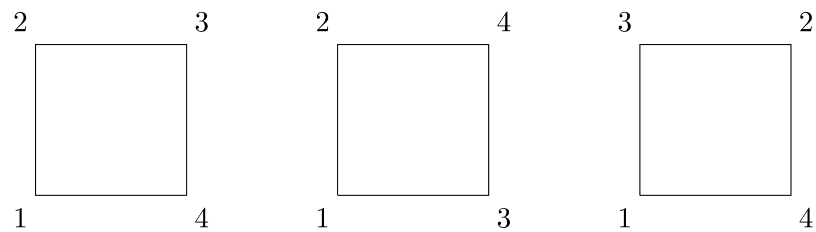
The only 3-Sylow subgroup of D_6 is $\{1, r^2, r^4\} = \langle r^2 \rangle$.

[1] Also called a Sylow p-subgroup. The term "p-Sylow subgroup" is used in Herstein's *Topics in Algebra* (2nd ed.), which is where I first learned group theory.

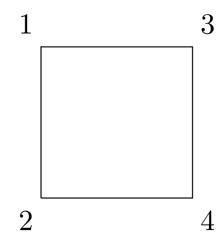
In a group of order 24, a 2-Sylow subgroup has order 8 and a 3-Sylow subgroup has order 3. Let's look at two examples.

Example 1.5. In S_4, the 3-Sylow subgroups are the 3-Sylow subgroups of A_4 (an element of 3-power order in S_4 must be a 3-cycle, and they all lie in A_4). We determined the 3-Sylow subgroups of A_4 in Example 1.3; there are four of them.

There are three 2-Sylow subgroups of S_4, and they are interesting to work out since they can be understood as *copies of* D_4 *inside* S_4. The number of ways to label the four vertices of a square as 1, 2, 3, and 4 is $4! = 24$, but up to rotations and reflections of the square there are really just three different ways of carrying out the labeling, as follows.



Every other labeling of the square is a rotated or reflected version of one of these three squares. For example, the square below is obtained from the middle square above by reflecting across a horizontal line through the middle of the square.



When D_4 acts on a square with labeled vertices, each motion of D_4 creates a permutation of the four vertices, and this permutation is an element of S_4. For example, a 90-degree rotation of the square is a 4-cycle on the vertices. In this way we obtain a copy of D_4 inside S_4. The three essentially different labelings of the vertices of the square above embed D_4 into S_4 as three different subgroups of order 8:

$$\{1,\ (1234),\ (1432),\ (12)(34),\ (13)(24),\ (14)(23),\ (13),\ (24)\} = \langle (1234), (13)\rangle,$$

$$\{1,\ (1243),\ (1342),\ (12)(34),\ (13)(24),\ (14)(23),\ (14),\ (23)\} = \langle (1243), (14)\rangle,$$

$$\{1,\ (1324),\ (1423),\ (12)(34),\ (13)(24),\ (14)(23),\ (12),\ (34)\} = \langle (1324), (12)\rangle.$$

These are the 2-Sylow subgroups of S_4.

Example 1.6. The group $\mathrm{SL}_2(\mathbf{Z}/(3))$ has order 24. It is not isomorphic to S_4 since its center $\{\pm I_2\}$ is nontrivial. By explicit calculation, $\mathrm{SL}_2(\mathbf{Z}/(3))$ has only 8 elements with 2-power order:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

These form the only 2-Sylow subgroup, which is isomorphic to Q_8 by labeling the matrices in the first row as $1, i, j, k$ and the matrices in the second row as $-1, -i, -j, -k$.

There are four 3-Sylow subgroups of $\mathrm{SL}_2(\mathbf{Z}/(3))$: $\langle(\begin{smallmatrix}1&1\\0&1\end{smallmatrix})\rangle$, $\langle(\begin{smallmatrix}1&0\\1&1\end{smallmatrix})\rangle$, $\langle(\begin{smallmatrix}0&1\\2&2\end{smallmatrix})\rangle$, and $\langle(\begin{smallmatrix}0&2\\1&2\end{smallmatrix})\rangle$.

Here are the Sylow theorems. They are often given in three parts. The result we call Sylow III* is not always stated explicitly as part of the Sylow theorems.[2]

Theorem 1.7 (Sylow I). *A finite group G has a p-Sylow subgroup for every prime p and each p-subgroup of G lies in some p-Sylow subgroup of G.*

Theorem 1.8 (Sylow II). *For each prime p, the p-Sylow subgroups of G are conjugate.*

Theorem 1.9 (Sylow III). *For each prime p, let n_p be the number of p-Sylow subgroups of G. Write $|G| = p^k m$, where p doesn't divide m. Then*

$$n_p \mid m \quad \textit{and} \quad n_p \equiv 1 \bmod p.$$

Theorem 1.10 (Sylow III*). *For each prime p, let n_p be the number of p-Sylow subgroups of G. Then $n_p = [G : \mathrm{N}(P)]$, where P is a p-Sylow subgroup and $\mathrm{N}(P)$ is its normalizer.*

The existence part of Sylow I has been illustrated in all the previous examples.

Sylow II says for two p-Sylow subgroups H and K of G that there is some $g \in G$ such that $gHg^{-1} = K$. This is illustrated in the table below, where Example 1.2 is skipped since $\mathbf{Z}/(12)$ is abelian.

Example	Group	Size	p	H	K	g
1.3	A_4	12	3	$\langle(123)\rangle$	$\langle(124)\rangle$	(243)
1.4	D_6	12	2	$\langle r^3, s\rangle$	$\langle r^3, rs\rangle$	r^2
1.5	S_4	24	2	$\langle(1234), (13)\rangle$	$\langle(1243), (14)\rangle$	(34)
1.6	$\mathrm{SL}_2(\mathbf{Z}/(3))$	24	3	$\langle(\begin{smallmatrix}1&1\\0&1\end{smallmatrix})\rangle$	$\langle(\begin{smallmatrix}1&0\\1&1\end{smallmatrix})\rangle$	$(\begin{smallmatrix}0&1\\2&1\end{smallmatrix})$

When trying to conjugate one cyclic subgroup to another cyclic subgroup, be careful: not all generators of the two groups have to be conjugate. For example, in A_4 the subgroups $\langle(123)\rangle = \{(1), (123), (132)\}$ and $\langle(124)\rangle = \{(1), (124), (142)\}$ are conjugate, but the conjugacy class of (123) in A_4 is $\{(123), (142), (134), (243)\}$, so there's no way to conjugate (123) to (124) by an element of A_4; we must conjugate (123) to (142). The 3-cycles (123) and (124) are conjugate in S_4, but not in A_4. Similarly, $(\begin{smallmatrix}1&1\\0&1\end{smallmatrix})$ and $(\begin{smallmatrix}1&0\\1&1\end{smallmatrix})$ are conjugate in $\mathrm{GL}_2(\mathbf{Z}/(3))$ but not in $\mathrm{SL}_2(\mathbf{Z}/(3))$, so when Sylow II says the subgroups $\langle(\begin{smallmatrix}1&1\\0&1\end{smallmatrix})\rangle$ and $\langle(\begin{smallmatrix}1&0\\1&1\end{smallmatrix})\rangle$ are conjugate in $\mathrm{SL}_2(\mathbf{Z}/(3))$ a conjugating matrix must send $(\begin{smallmatrix}1&1\\0&1\end{smallmatrix})$ to $(\begin{smallmatrix}1&0\\1&1\end{smallmatrix})^2 = (\begin{smallmatrix}1&0\\2&1\end{smallmatrix})$.

Let's see what Sylow III tells us about the number of 2-Sylow and 3-Sylow subgroups of a group of order 12. For $p = 2$ and $p = 3$ in Sylow III, the divisibility conditions are $n_2 \mid 3$ and $n_3 \mid 4$ and the congruence conditions are $n_2 \equiv 1 \bmod 2$ and $n_3 \equiv 1 \bmod 3$. The divisibility conditions imply n_2 is 1 or 3 and n_3 is 1, 2, or 4. The congruence $n_2 \equiv 1 \bmod 2$ tells us nothing new (1 and 3 are both odd), but the congruence $n_3 \equiv 1 \bmod 3$ rules out the option $n_3 = 2$. Therefore n_2 is 1 or 3 and n_3 is 1 or 4 when $|G| = 12$.

If $|G| = 24$ we again find n_2 is 1 or 3 while n_3 is 1 or 4. (For instance, from $n_3 \mid 8$ and $n_3 \equiv 1 \bmod 3$ the only choices are $n_3 = 1$ and $n_3 = 4$.) Therefore as soon as we find more than one 2-Sylow subgroup there must be three of them, and as soon as we find more than one 3-Sylow subgroup there must be four of them. The table below shows the values of n_2 and n_3 in the examples above.

[2] In Sylow's paper, parts I and III* are in [4, Théorème I] while parts II and III are in [4, Théorème II].

Example	Group	Size	n_2	n_3
1.2	$\mathbf{Z}/(12)$	12	1	1
1.3	A_4	12	1	4
1.4	D_6	12	3	1
1.5	S_4	24	3	4
1.6	$\mathrm{SL}_2(\mathbf{Z}/(3))$	24	1	4

2. Proof of the Sylow Theorems

Our proof of the Sylow theorems will use group actions. The table below is a summary. For each theorem the table lists a group, a set it acts on, and the action. Let $\mathrm{Syl}_p(G)$ be the set of p-Sylow subgroups of G, so $n_p = |\,\mathrm{Syl}_p(G)|$.

Theorem	Group	Set	Action
Sylow I	p-subgroup H	G/H	left mult.
Sylow II	p-Sylow subgroup Q	G/P	left mult.
Sylow III ($n_p \mid m$)	G	$\mathrm{Syl}_p(G)$	conjugation
Sylow III ($n_p \equiv 1 \bmod p$)	$P \in \mathrm{Syl}_p(G)$	$\mathrm{Syl}_p(G)$	conjugation
Sylow III*	G	$\mathrm{Syl}_p(G)$	conjugation

The two conclusions of Sylow III are listed separately in the table since they are proved using different group actions.

Our proofs will usually involve the action of a p-group on a set and use the fixed-point congruence for such actions: when X is a finite set being acted on by a finite p-group Γ,

$$|X| \equiv |\,\mathrm{Fix}_\Gamma(X)| \bmod p, \tag{2.1}$$

where $\mathrm{Fix}_\Gamma(X)$ is the set of fixed points of Γ in X.

Proof of Sylow I: Let p^k be the highest power of p in $|G|$. The result is obvious if $k = 0$, since the trivial subgroup is a p-Sylow subgroup, so we can take $k \geq 1$, hence $p \mid |G|$.

Our strategy for proving Sylow I is to **prove a stronger result**: *G has a subgroup of order p^i for $0 \leq i \leq k$.* More precisely, if $|H| = p^i$ and $i < k$, we will show there is a p-subgroup $H' \supset H$ with $[H' : H] = p$, so $|H'| = p^{i+1}$. Then, starting with H as the trivial subgroup, repeat this process with H' in place of H to create larger subgroups

$$\{e\} = H_0 \subset H_1 \subset H_2 \subset \cdots$$

with $|H_i| = p^i$, and after k steps we reach H_k, of order p^k, which is a p-Sylow subgroup. Starting with H as a p-subgroup, we will have shown H is contained in a p-Sylow subgroup.

Consider the left multiplication action of H on the left cosets G/H (note G/H might not be a group). This is an action of a finite p-group H on the set G/H, so by the fixed-point congruence (2.1) for actions of nontrivial p-groups,

$$|G/H| \equiv |\,\mathrm{Fix}_H(G/H)| \bmod p. \tag{2.2}$$

Here is what it means for gH in G/H to be fixed by the group H acting by left multiplication:

$$\begin{aligned} hgH = gH \text{ for all } h \in H \iff & \; hg \in gH \text{ for all } h \in H \\ \iff & \; g^{-1}hg \in H \text{ for all } h \in H \\ \iff & \; g^{-1}Hg \subset H \\ \iff & \; g^{-1}Hg = H \text{ because } |g^{-1}Hg| = |H| \\ \iff & \; g \in \mathrm{N}(H). \end{aligned}$$

Thus $\mathrm{Fix}_H(G/H) = \{gH : g \in \mathrm{N}(H)\} = \mathrm{N}(H)/H$, so (2.2) becomes

$$[G : H] \equiv [\mathrm{N}(H) : H] \bmod p. \tag{2.3}$$

Because $H \lhd \mathrm{N}(H)$, $\mathrm{N}(H)/H$ *is a group.*

When $|H| = p^i$ and $i < k$, the index $[G : H]$ is divisible by p, so the congruence (2.3) implies $[\mathrm{N}(H) : H]$ is divisible by p. Therefore the *group* $\mathrm{N}(H)/H$ has order divisible by p. By Cauchy's theorem, $\mathrm{N}(H)/H$ has a subgroup $\langle \overline{g} \rangle$ of order p (necessarily cyclic). The reduction homomorphism $\mathrm{N}(H) \to \mathrm{N}(H)/H$ has kernel H, so the inverse image of $\langle \overline{g} \rangle$ in $\mathrm{N}(H)$ is a subgroup that contains H and has order $p|H| = p^{i+1}$.[3] We have shown each subgroup of G with order p^i for $i < k$ is contained in a subgroup of G with order p^{i+1}. This can be repeated until we reach a subgroup of order p^k.

Proof of Sylow II: For p-Sylow subgroups P and Q, we want to show they are conjugate. Let Q act on G/P by left multiplication. Since Q is a finite p-group, (2.1) says

$$|G/P| \equiv |\mathrm{Fix}_Q(G/P)| \bmod p.$$

The left side is $[G : P]$, which is nonzero modulo p since P is a p-Sylow subgroup. Thus $|\mathrm{Fix}_Q(G/P)|$ can't be 0, so there is a fixed point in G/P. Call it gP. That is, $qgP = gP$ for all $q \in Q$. Equivalently, $qg \in gP$ for all $q \in Q$, so $Q \subset gPg^{-1}$. Therefore $Q = gPg^{-1}$, since Q and gPg^{-1} have the same size and we're done.

Proof of Sylow III: We will prove $n_p \equiv 1 \bmod p$ and then $n_p \mid m$.

To show $n_p \equiv 1 \bmod p$, let P act on $\mathrm{Syl}_p(G)$ by conjugation. The size of $\mathrm{Syl}_p(G)$ is n_p. Since P is a finite p-group, (2.1) says

$$n_p \equiv |\{\text{fixed points}\}| \bmod p.$$

Fixed points for P acting by conjugation on $\mathrm{Syl}_p(G)$ are $Q \in \mathrm{Syl}_p(G)$ such that $gQg^{-1} = Q$ for all $g \in P$. One choice for Q is P. For all such Q, $P \subset \mathrm{N}(Q)$. Also $Q \subset \mathrm{N}(Q)$, so P and Q are p-Sylow subgroups in $\mathrm{N}(Q)$. Applying Sylow II to the group $\mathrm{N}(Q)$, P and Q are conjugate in $\mathrm{N}(Q)$. Since $Q \lhd \mathrm{N}(Q)$, the only subgroup of $\mathrm{N}(Q)$ conjugate to Q is Q, so $P = Q$. Thus P is the only fixed point when P acts on $\mathrm{Syl}_p(G)$, so $n_p \equiv 1 \bmod p$.

To show $n_p \mid m$, consider the action of G by conjugation on $\mathrm{Syl}_p(G)$. Since the p-Sylow subgroups are conjugate to each other (Sylow II), there is one orbit. A set on which a group acts with one orbit has size dividing the size of the group, so $n_p \mid |G|$. From $n_p \equiv 1 \bmod p$, the number n_p is relatively prime to p, so $n_p \mid m$ and we're done.

Proof of Sylow III*: Let P be a p-Sylow subgroup of G and let G act on $\mathrm{Syl}_p(G)$ by conjugation. By the orbit-stabilizer formula,

$$n_p = |\mathrm{Syl}_p(G)| = [G : \mathrm{Stab}_{\{P\}}].$$

The stabilizer $\mathrm{Stab}_{\{P\}}$ of the "point" P in $\mathrm{Syl}_p(G)$ (viewing P as a point is why we write $\{P\}$) is

$$\mathrm{Stab}_{\{P\}} = \{g : gPg^{-1} = P\} = \mathrm{N}(P).$$

Thus $n_p = [G : \mathrm{N}(P)]$ and we're done.

[3]If $f : G \to \widetilde{G}$ is a surjective homomorphism of finite groups with kernel K and M is a subgroup of $\widetilde{G}$, then the inverse image $f^{-1}(M) = \{g \in G : f(g) \in M\}$ is a subgroup of G and $|f^{-1}(M)| = |M||K|$. The reason $|f^{-1}(M)| = |M||K|$ is that $K \subset f^{-1}(M)$ so the restriction of f to a function $f^{-1}(M) \to M$ is a surjective homomorphism with kernel K, so $f^{-1}(M)/K \cong M$. Thus $|f^{-1}(M)|/|K| = |M|$, so $|f^{-1}(M)| = |M||K|$.

In the proof of Sylow I, we saw that if H is a p-subgroup of G that is not a p-Sylow subgroup then $\mathrm{N}(H)$ is strictly larger than H. What can be said about $\mathrm{N}(P)$ when P is a p-Sylow subgroup? It may or may not be larger than P, but we will show that taking the normalizer a second time will not give anything new.

Theorem 2.1. *Let P be a p-Sylow subgroup of a finite group G. Then $\mathrm{N}(\mathrm{N}(P)) = \mathrm{N}(P)$. More generally, if H is a subgroup of G that contains $\mathrm{N}(P)$ then $\mathrm{N}(H) = H$.*

Proof. We will prove $H \subset \mathrm{N}(H)$ and $\mathrm{N}(H) \subset H$. The containment $H \subset \mathrm{N}(H)$ is easy,

To prove $\mathrm{N}(H) \subset H$ let $x \in \mathrm{N}(H)$, so $xHx^{-1} = H$. Since $P \subset \mathrm{N}(P) \subset H$ we have $xPx^{-1} \subset xHx^{-1} = H$, so P and xPx^{-1} are both p-Sylow subgroups of H. By Sylow II for the group H, there is $y \in H$ such that $xPx^{-1} = yPy^{-1}$. Thus $y^{-1}xP(y^{-1}x)^{-1} = P$, so $y^{-1}x \in \mathrm{N}(P) \subset H$, so $x \in yH = H$. □

3. Historical Remarks

Sylow's proof of his theorems appeared in [4]. Here is what he showed about every prime p and finite group G (of course, without using the label "Sylow subgroup").

(1) There is a p-Sylow subgroup of G. Moreover, $[G : \mathrm{N}(P)] \equiv 1 \bmod p$ for each p-Sylow subgroup P.
(2) Let P be a p-Sylow subgroup of G. The number of p-Sylow subgroups of G is $[G : \mathrm{N}(P)]$. All p-Sylow subgroups of G are conjugate.
(3) Each finite p-group H with size p^k contains an increasing chain of subgroups
$$\{e\} = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_k \subset H,$$
where each subgroup has index p in the next one. In particular, $|H_i| = p^i$ for all i.

To prove part (3), which is [4, Théorème III], Sylow proved that every nontrivial p-group has a nontrivial center [4, p. 588]. Whle these results on finite p-groups appear in all books on group theory, that they are due to Sylow has been forgotten. If in (3) we take for H a p-Sylow subgroup of G then (3) shows G has a subgroup of order p^i for each p-power p^i dividing $|G|$, a result that is the second sentence of Sylow's paper.

Here is how Sylow [4, Théorème I] wrote item (1) above:[4]

> Si p^α désigne la plus grande puissance du nombre premier p qui divise l'ordre du groupe G, ce groupe contient un autre H de l'ordre p^α; si de plus $p^\alpha\nu$ désigne l'ordre du plus grand groupe contenu dans G dont les substitutions sont permutables à H, l'ordre de G sera de la forme $p^\alpha\nu(pm+1)$.

In English, using current terminology, this says

> If p^α is the largest power of the prime p which divides the size of the group G, this group contains a subgroup H of order p^α; if moreover $p^\alpha\nu$ is the size of the largest subgroup of G that normalizes H, the size of G is of the form $p^\alpha\nu(pm+1)$.

Sylow did not have the abstract concept of a group: all groups for him arose as subgroups of symmetric groups, so groups were always "groupes de substitutions." The condition that an element $x \in G$ is "permutable" with a subgroup H means $xH = Hx$, or in other words $x \in \mathrm{N}(H)$. The end of the first part of his theorem says the normalizer of a Sylow subgroup has index $pm + 1$ for some m, which means the index is $\equiv 1 \bmod p$.

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[4]We modify some of his notation: he wrote the subgroup as g, not H, and the prime as n, not p.

Sylow's approach to proving the existence of Sylow subgroups of G was to start with a p-subgroup H in G of maximal order and prove $p \nmid [G : H]$, so $|H|$ is the largest power of p dividing $|G|$. (A version of his proof in modern language is in [1, Sect. 2] and [5, Sect. 2], or Section 2 in https://kconrad.math.uconn.edu/blurbs/grouptheory/sylowmore.pdf.) In some accounts of the Sylow theorems, a p-Sylow subgroup of G is not defined as a subgroup of G whose order is the biggest power of p dividing $|G|$, but as a p-subgroup of G with maximal order. That point of view goes right back to Sylow's own work.

4. Analogues of the Sylow Theorems

There are analogues of the first two Sylow theorems and Theorem 2.1 for certain types of subgroups.

(1) A *Hall subgroup* of a finite group G is a subgroup H whose order and index are relatively prime. For example, in a group of order 60 a subgroup of order 12 has index 5 and thus is a Hall subgroup. Every Sylow subgroup is a Hall subgroup, and a p-subgroup is a Hall subgroup if and only if it is a Sylow subgroup. Hall subgroups were introduced in 1928 by the group theorist Philip Hall [2], who called them S subgroups rather than Hall subgroups. He proved that in every *solvable* group of order n and each d dividing n such that $(d, n/d) = 1$ there is a Hall subgroup of order d, all Hall subgroups with the same order are conjugate, and the normalizer of a Hall subgroup of a solvable group is its own normalizer. About 10 years later, Hall [3] proved a converse result showing the solvability hypothesis on G is necessary: if a finite group G of order n contains a Hall subgroup of order d for each d dividing n such that $(d, n/d) = 1$, then G is solvable.

(2) In a compact connected Lie group G, *maximal tori* (maximal connected abelian subgroups of G) satisfy properties analogous to Sylow subgroups: they exist, every torus is in a maximal torus, and all maximal tori are conjugate. The proof of conjugacy uses the Lefschetz fixed point theorem. (This plays a role analogous to the the fixed-point congruence (2.1) in the proof of the Sylow theorems.) Like normalizers of Sylow subgroups, the normalizer of a maximal torus is its own normalizer. Unlike the relation of Sylow subgroups in a general finite group, maximal tori are always abelian and every element of G is in some maximal torus.

(3) In a connected linear algebraic group, *maximal connected unipotent subgroups* are like Sylow subgroups: they exist, every connected unipotent subgroup is in a maximal connected unipotent subgroup, and all maximal connected unipotent subgroups are conjugate. The proof of conjugacy uses the Borel fixed point theorem. The normalizer of a maximal connected unipotent subgroup is called a Borel subgroup, and like normalizers of Sylow subgroups each Borel subgroup is its own normalizer.

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