THE SYLOW THEOREMS

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1. Introduction

The converse of Lagrange’s theorem is false: if $G$ is a finite group and $d \mid |G|$, then there may not be a subgroup of $G$ with order $d$. The simplest example of this is the group $A_4$, of order 12, which has no subgroup of order 6. The Norwegian mathematician Peter Ludwig Sylow [1] discovered that a converse result is true when $d$ is a prime power: if $p$ is a prime number and $p^k \mid |G|$ then $G$ must contain a subgroup of order $p^k$. Sylow also discovered important relations among the subgroups with order the largest power of $p$ dividing $|G|$, such as the fact that all subgroups of that order are conjugate to each other.

For example, a group of order $100 = 2^2 \cdot 5^2$ must contain subgroups of order 1, 2, 4, 5, and 25, the subgroups of order 4 are conjugate to each other, and the subgroups of order 25 are conjugate to each other. It is not necessarily the case that the subgroups of order 2 are conjugate or that the subgroups of order 5 are conjugate.

Definition 1.1. Let $G$ be a finite group and $p$ be a prime. Any subgroup of $G$ whose order is the highest power of $p$ dividing $|G|$ is called a $p$-Sylow subgroup of $G$. A $p$-Sylow subgroup for some $p$ is called a Sylow subgroup.

In a group of order 100, a 2-Sylow subgroup has order 4, a 5-Sylow subgroup has order 25, and a $p$-Sylow subgroup is trivial if $p \neq 2$ or 5.

In a group of order 12, a 2-Sylow subgroup has order 4, a 3-Sylow subgroup has order 3, and a $p$-Sylow subgroup is trivial if $p > 3$. Let’s look at a few examples of Sylow subgroups in groups of order 12.

Example 1.2. In $\mathbb{Z}/(12)$, the only 2-Sylow subgroup is $\{0, 3, 6, 9\} = \langle 3 \rangle$ and the only 3-Sylow subgroup is $\{0, 4, 8\} = \langle 4 \rangle$.

Example 1.3. In $A_4$ there is one subgroup of order 4, so the only 2-Sylow subgroup is

$$\{(1), (12)(34), (13)(24), (14)(23)\} = \langle (12)(34), (14)(23) \rangle.$$

There are four 3-Sylow subgroups:

$$\{(1), (123), (132)\} = \langle (123) \rangle, \quad \{(1), (124), (142)\} = \langle (124) \rangle,$$

$$\{(1), (134), (143)\} = \langle (134) \rangle, \quad \{(1), (234), (243)\} = \langle (234) \rangle.$$

Example 1.4. In $D_6$ there are three 2-Sylow subgroups:

$$\{1, r^3, s, r^3s\} = \langle r^3, s \rangle, \quad \{1, r^3, rs, r^4s\} = \langle r^3, rs \rangle, \quad \{1, r^3, r^2s, r^5s\} = \langle r^3, r^2s \rangle.$$

The only 3-Sylow subgroup of $D_6$ is $\{1, r^2, r^4\} = \langle r^2 \rangle$.

In a group of order 24, a 2-Sylow subgroup has order 8 and a 3-Sylow subgroup has order 3. Let’s look at two examples.
Example 1.5. In $S_4$, the 3-Sylow subgroups are the 3-Sylow subgroups of $A_4$ (an element of 3-power order in $S_4$ must be a 3-cycle, and they all lie in $A_4$). We determined the 3-Sylow subgroups of $A_4$ in Example 1.3; there are four of them.

There are three 2-Sylow subgroups of $S_4$, and they are interesting to work out since they can be understood as copies of $D_4$ inside $S_4$. The number of ways to label the four vertices of a square as 1, 2, 3, and 4 is $4! = 24$, but up to rotations and reflections of the square there are really just three different ways of carrying out the labeling, as follows.

Any other labeling of the square is a rotated or reflected version of one of these three squares. For example, the square below is obtained from the middle square above by reflecting across a horizontal line through the middle of the square.

When $D_4$ acts on a square with labeled vertices, each motion of $D_4$ creates a permutation of the four vertices, and this permutation is an element of $S_4$. For example, a 90 degree rotation of the square is a 4-cycle on the vertices. In this way we obtain a copy of $D_4$ inside $S_4$. The three essentially different labelings of the vertices of the square above embed $D_4$ into $S_4$ as three different subgroups of order 8:

$\{1, (12)(34), (13)(24), (14)(23), (12), (13), (24)\} = \langle (1234), (13) \rangle$,

$\{1, (1243), (1342), (12)(34), (13)(24), (14)(23), (14), (23)\} = \langle (1243), (14) \rangle$,

$\{1, (1324), (1423), (12)(34), (13)(24), (14)(23), (12), (34)\} = \langle (1324), (12) \rangle$.

These are the 2-Sylow subgroups of $S_4$.

Example 1.6. The group $\text{SL}_2(\mathbb{Z}/(3))$ has order 24. An explicit tabulation of the elements of this group reveals that there are only 8 elements in the group with 2-power order:

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$

These form the only 2-Sylow subgroup, which is isomorphic to $Q_8$ by labeling the matrices in the first row as $1, i, j, k$ and the matrices in the second row as $-1, -i, -j, -k$.

There are four 3-Sylow subgroups: $\langle (\frac{1}{0}, 1 \rangle, \langle (\frac{1}{1}, 1 \rangle, \langle (\frac{0}{1}, 2 \rangle, and \langle (\frac{2}{1}, 2 \rangle).$
Here are the Sylow theorems. They are often given in three parts. The result we call Sylow III* is not always stated explicitly as part of the Sylow theorems.

**Theorem 1.7** (Sylow I). A finite group $G$ has a $p$-Sylow subgroup for every prime $p$ and each $p$-subgroup of $G$ lies in some $p$-Sylow subgroup of $G$.

**Theorem 1.8** (Sylow II). For each prime $p$, the $p$-Sylow subgroups of $G$ are conjugate.

**Theorem 1.9** (Sylow III). For each prime $p$, let $n_p$ be the number of $p$-Sylow subgroups of $G$. Write $|G| = p^k m$, where $p$ doesn’t divide $m$. Then

$$n_p \equiv 1 \mod p \quad \text{and} \quad n_p \mid m.$$  

**Theorem 1.10** (Sylow III*). For each prime $p$, let $n_p$ be the number of $p$-Sylow subgroups of $G$. Then $n_p = [G : N(P)]$, where $P$ is a $p$-Sylow subgroup and $N(P)$ is its normalizer.

The existence part of Sylow I has been illustrated in all the previous examples. Sylow II says for two $p$-Sylow subgroups $H$ and $K$ of $G$ that there is some $g \in G$ such that $gHg^{-1} = K$. This is illustrated in the table below, where Example 1.2 is skipped since $\mathbb{Z}/(12)$ is abelian.

<table>
<thead>
<tr>
<th>Example</th>
<th>Group</th>
<th>Size</th>
<th>$p$</th>
<th>$H$</th>
<th>$K$</th>
<th>$g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3</td>
<td>$A_4$</td>
<td>12</td>
<td>3</td>
<td>$\langle(123)\rangle$</td>
<td>$\langle(124)\rangle$</td>
<td>$(243)$</td>
</tr>
<tr>
<td>1.4</td>
<td>$D_6$</td>
<td>12</td>
<td>2</td>
<td>$\langle r^3, s \rangle$</td>
<td>$\langle r^3, rs \rangle$</td>
<td>$r^2$</td>
</tr>
<tr>
<td>1.5</td>
<td>$S_4$</td>
<td>24</td>
<td>2</td>
<td>$\langle(1234), (13)\rangle$</td>
<td>$\langle(1243), (14)\rangle$</td>
<td>$(34)$</td>
</tr>
<tr>
<td>1.6</td>
<td>$\text{SL}_2(\mathbb{Z}/(3))$</td>
<td>24</td>
<td>3</td>
<td>$\langle(1 \ 0) \ \rangle$</td>
<td>$\langle(1 \ 0) \ \rangle$</td>
<td>$\langle(0 \ 1) \ \rangle$</td>
</tr>
</tbody>
</table>

When trying to conjugate one cyclic subgroup to another cyclic subgroup, be careful: not all generators of the two groups have to be conjugate. For example, in $A_4$ the subgroups $\langle(123)\rangle = \{(1), (123), (132)\}$ and $\langle(124)\rangle = \{(1), (124), (142)\}$ are conjugate, but the conjugacy class of $(123)$ in $A_4$ is $\{(123), (142), (134), (243)\}$, so there’s no way to conjugate $(123)$ to $(124)$ by an element of $A_4$; we must conjugate $(123)$ to $(142)$. The 3-cycles $(123)$ and $(124)$ are conjugate in $S_4$, but not in $A_4$. Similarly, $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ and $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ are conjugate in $\text{GL}_2(\mathbb{Z}/(3))$ but not in $\text{SL}_2(\mathbb{Z}/(3))$, so when Sylow II says the subgroups $\langle(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \rangle$ and $\langle(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \rangle$ are conjugate in $\text{SL}_2(\mathbb{Z}/(3))$ a conjugating matrix must send $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ to $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})^2 = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$.

Let’s see what Sylow III tells us about the number of 2-Sylow and 3-Sylow subgroups of a group of order 12. For $p = 2$ and $p = 3$ in Sylow III, the divisibility conditions are $n_2 \mid 3$ and $n_3 \mid 4$ and the congruence conditions are $n_2 \equiv 1 \mod 2$ and $n_3 \equiv 1 \mod 3$. The divisibility conditions imply $n_2$ is 1 or 3 and $n_3$ is 1, 2, or 4. The congruence $n_2 \equiv 1 \mod 2$ tells us nothing new (1 and 3 are both odd), but the congruence $n_3 \equiv 1 \mod 3$ rules out the option $n_3 = 2$. Therefore $n_2$ is 1 or 3 and $n_3$ is 1 3 or 4 when $|G| = 12$. If $|G| = 24$ we again find $n_2$ is 1 or 3 while $n_3$ is 1 or 4. (For instance, from $n_3 \mid 8$ and $n_3 \equiv 1 \mod 3$ the only choices are $n_3 = 1$ and $n_3 = 4$.) Therefore as soon as we find more than one 2-Sylow subgroup there must be three of them, and as soon as we find more than one 3-Sylow subgroup there must be four of them. The table below shows the values of $n_2$ and $n_3$ in the examples above.

<table>
<thead>
<tr>
<th>Example</th>
<th>Group</th>
<th>Size</th>
<th>$n_2$</th>
<th>$n_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>$\mathbb{Z}/(12)$</td>
<td>12</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1.3</td>
<td>$A_4$</td>
<td>12</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>1.4</td>
<td>$D_6$</td>
<td>12</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>1.5</td>
<td>$S_4$</td>
<td>24</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1.6</td>
<td>$\text{SL}_2(\mathbb{Z}/(3))$</td>
<td>24</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>
2. Proof of the Sylow Theorems

Our proof of the Sylow theorems will use group actions, which we assume the reader knows. The table below is a summary. For each theorem the table lists a group, a set it acts on, and the action. Let \( \text{Syl}_p(G) \) be the set of \( p \)-Sylow subgroups of \( G \), so \( n_p = |\text{Syl}_p(G)| \).

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Group</th>
<th>Set</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sylow I</td>
<td>( p )-subgroup ( H )</td>
<td>( G/H )</td>
<td>left mult.</td>
</tr>
<tr>
<td>Sylow II</td>
<td>( p )-Sylow subgroup ( Q )</td>
<td>( G/P )</td>
<td>left mult.</td>
</tr>
<tr>
<td>Sylow III (( n_p \equiv 1 \mod p ))</td>
<td>( P \in \text{Syl}_p(G) )</td>
<td>( \text{Syl}_p(G) )</td>
<td>conjugation</td>
</tr>
<tr>
<td>Sylow III (( n_p \mid m ))</td>
<td>( G )</td>
<td>( \text{Syl}_p(G) )</td>
<td>conjugation</td>
</tr>
<tr>
<td>Sylow III′</td>
<td>( G )</td>
<td>( \text{Syl}_p(G) )</td>
<td>conjugation</td>
</tr>
</tbody>
</table>

The two conclusions of Sylow III are listed separately in the table since they are proved using different group actions.

Our proofs will usually involve the action of a \( p \)-group on a set and use the fixed-point congruence for such actions: \( |X| \equiv |\text{Fix}_\Gamma(X)| \mod p \), where \( X \) is a finite set being acted on by a finite \( p \)-group \( \Gamma \) and \( \text{Fix}_\Gamma(X) \) is the fixed points of \( \Gamma \) in \( X \).

**Proof of Sylow I:** Let \( p^k \) be the highest power of \( p \) in \( |G| \). The result is obvious if \( k = 0 \), since the trivial subgroup is a \( p \)-Sylow subgroup, so we can take \( k \geq 1 \), hence \( p \mid |G| \).

Our strategy for proving Sylow I is to prove a stronger result: there is a subgroup of order \( p^i \) for \( 0 \leq i \leq k \). More specifically, if \( |H| = p^i \) and \( i < k \), we will show there is a \( p \)-subgroup \( H' \supset H \) with \( |H' : H| = p \), so \( |H'| = p^{i+1} \). Then, starting with \( H \) as the trivial subgroup, we can repeat this process with \( H' \) in place of \( H \) to create increasingly larger subgroups

\[
\{e\} = H_0 \subset H_1 \subset H_2 \subset \cdots
\]

where \( |H_i| = p^i \), and after \( k \) steps we reach \( H_k \), which is a \( p \)-Sylow subgroup of \( G \). And if we start with \( H \) as a \( p \)-subgroup, we will have shown \( H \) is contained in a \( p \)-Sylow subgroup.

Consider the left multiplication action of \( H \) on the left cosets \( G/H \) (this need not be a group). This is an action of a finite \( p \)-group \( H \) on the set \( G/H \), so by the fixed-point congruence for actions of nontrivial \( p \)-groups,

\[
|G/H| \equiv |\text{Fix}_H(G/H)| \mod p.
\]

Here is what it means for \( gH \) in \( G/H \) to be fixed by the group \( H \) acting by left multiplication:

\[
hgH = gH \text{ for all } h \in H \quad \iff \quad hg \in gH \text{ for all } h \in H
\]

\[
\iff \quad g^{-1}hg \in H \text{ for all } h \in H
\]

\[
\iff \quad g^{-1}Hg \subset H
\]

\[
\iff \quad g^{-1}Hg = H \text{ because } |g^{-1}Hg| = |H|
\]

\[
\iff \quad g \in \text{N}(H).
\]

Thus \( \text{Fix}_H(G/H) = \{gH : g \in \text{N}(H)\} = \text{N}(H)/H \), so (2.1) becomes

\[
|G : H| \equiv |\text{N}(H) : H| \mod p.
\]

Because \( H \triangleleft \text{N}(H) \), \( \text{N}(H)/H \) is a group.

When \( |H| = p^i \) and \( i < k \), the index \( |G : H| \) is divisible by \( p \), so the congruence (2.2) implies \( |\text{N}(H) : H| \) is divisible by \( p \), so \( \text{N}(H)/H \) is a group with order divisible by \( p \). Thus \( \text{N}(H)/H \) has a subgroup of order \( p \) by Cauchy’s theorem. All subgroups of the quotient group \( \text{N}(H)/H \) have the form \( H'/H \), where \( H' \) is a subgroup between \( H \) and
N(H). Therefore a subgroup of order \( p \) in \( N(H)/H \) is \( H'/H \) such that \( [H' : H] = p \), so \( |H'| = p|H| = p^{i+1} \). This can be repeated until we reach a subgroup of order \( p^k \), and we’re done.

**Proof of Sylow II:** For \( p \)-Sylow subgroups \( P \) and \( Q \), we want to show they are conjugate. Consider the action of \( Q \) on \( G/P \) by left multiplication. Since \( Q \) is a finite \( p \)-group,

\[
|G/P| \equiv |\text{Fix}_Q(G/P)| \mod p.
\]

The left side is \( [G : P] \), which is nonzero modulo \( p \) since \( P \) is a \( p \)-Sylow subgroup. Thus \( |\text{Fix}_Q(G/P)| \) can’t be 0, so there is a fixed point in \( G/P \). Call it \( gP \). That is, \( gQ = gP \) for all \( g \in Q \). Equivalently, \( gQg^{-1} = Q \) for all \( g \in Q \), so \( Q \subseteq gPg^{-1} \). Therefore \( Q = gPg^{-1} \), since \( Q \) and \( gPg^{-1} \) have the same size and we’re done.

**Proof of Sylow III:** We will prove \( n_p \equiv 1 \mod p \) and then \( n_p \mid m \).

To show \( n_p \equiv 1 \mod p \), consider the action of \( P \) on the set \( \text{Syl}_p(G) \) by conjugation. The size of \( \text{Syl}_p(G) \) is \( n_p \). Since \( P \) is a finite \( p \)-group, by the fixed-point congruence we have

\[
n_p \equiv |\{\text{fixed points}\}| \mod p.
\]

Fixed points for \( P \) acting by conjugation on \( \text{Syl}_p(G) \) are \( Q \in \text{Syl}_p(G) \) such that \( gQg^{-1} = Q \) for all \( g \in P \). One choice for \( Q \) is \( P \). For all such \( Q \), \( P \subseteq N(Q) \). Also \( Q \subseteq N(Q) \), so \( P \) and \( Q \) are \( p \)-Sylow subgroups in \( N(Q) \). Applying Sylow II to the group \( N(Q) \), \( P \) and \( Q \) are conjugate in \( N(Q) \). Since \( Q \triangleleft N(Q) \), the only subgroup of \( N(Q) \) conjugate to \( Q \) is \( Q \), so \( P = Q \). Thus \( P \) is the only fixed point when \( P \) acts on \( \text{Syl}_p(G) \), so \( n_p \equiv 1 \mod p \).

To show \( n_p \mid m \), consider the action of \( G \) by conjugation on \( \text{Syl}_p(G) \). Since the \( p \)-Sylow subgroups are conjugate to each other (Sylow II), there is one orbit. A set on which a group acts with one orbit has size dividing the size of the group, so \( n_p \mid |G| \). From \( n_p \equiv 1 \mod p \), the number \( n_p \) is relatively prime to \( p \), so \( n_p \mid m \) and we’re done.

**Proof of Sylow III:** Let \( P \) be a \( p \)-Sylow subgroup of \( G \) and let \( G \) act on \( \text{Syl}_p(G) \) by conjugation. By the orbit-stabilizer formula,

\[
n_p = |\text{Syl}_p(G)| = |G : \text{Stab}_p|.
\]

The stabilizer \( \text{Stab}_p \) of the “point” \( P \) in \( \text{Syl}_p(G) \) (viewing \( P \) as a point is why we write \( \{P\} \)) is

\[
\text{Stab}_p = \{g : gPg^{-1} = P\} = N(P).
\]

Thus \( n_p = |G : N(P)| \) and we’re done.

In the proof of Sylow I, we saw that if \( H \) is a \( p \)-subgroup of \( G \) that is not a \( p \)-Sylow subgroup then \( N(H) \) is strictly larger than \( H \). What can be said about \( N(P) \) when \( P \) is a \( p \)-Sylow subgroup? It may or may not be larger than \( P \), but we will show that taking the normalizer a second time will not give anything new.

**Theorem 2.1.** Let \( P \) be a \( p \)-Sylow subgroup of a finite group \( G \). Then \( N(N(P)) = N(P) \). More generally, if \( H \) is a subgroup of \( G \) that contains \( N(P) \) then \( N(H) = H \).

**Proof.** We will prove \( H \subseteq N(H) \) and \( N(H) \subseteq H \). The containment \( H \subseteq N(H) \) is easy.

To prove \( N(H) \subseteq H \) let \( x \in N(H) \), so \( xHx^{-1} = H \). Since \( P \subseteq N(P) \subseteq H \) we have \( xPx^{-1} \subseteq xHx^{-1} = H \), so \( P \) and \( xPx^{-1} \) are both \( p \)-Sylow subgroups of \( H \). By Sylow II for the group \( H \), there is \( y \in H \) such that \( xPx^{-1} = yPy^{-1} \). Thus \( y^{-1}xP(y^{-1}x)^{-1} = P \), so \( y^{-1}x \in N(P) \subseteq H \), so \( x \in yH = H \). □
3. Historical Remarks

Sylow’s proof of his theorems appeared in [1]. Here is what he showed (of course, without using the label “Sylow subgroup”).

1. There exist \( p \)-Sylow subgroups. Moreover, \( [G : N(P)] \equiv 1 \mod p \) for each \( p \)-Sylow subgroup \( P \).
2. Let \( P \) be a \( p \)-Sylow subgroup. The number of \( p \)-Sylow subgroups is \( [G : N(P)] \). All \( p \)-Sylow subgroups are conjugate.
3. Any finite \( p \)-group \( G \) with size \( p^k \) contains an increasing chain of subgroups

\[
\{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_k \subset G,
\]

where each subgroup has index \( p \) in the next one. In particular, \( |G_i| = p^i \) for all \( i \).

Here is how Sylow phrased his first theorem (the first item on the above list):

Si \( p^\alpha \) désigne la plus grande puissance du nombre premier \( p \) qui divise l’ordre du groupe \( G \), ce groupe contient un autre \( H \) de l’ordre \( p^\alpha \); si de plus \( p^\alpha \nu \) désigne l’ordre du plus grand groupe contenu dans \( G \) dont les substitutions sont permutables à \( H \), l’ordre de \( G \) sera de la forme \( p^\alpha \nu(pm + 1) \).

In English, using current terminology, this says

If \( p^\alpha \) is the largest power of the prime \( p \) which divides the size of the group \( G \), this group contains a subgroup \( H \) of order \( p^\alpha \); if moreover \( p^\alpha \nu \) is the size of the largest subgroup of \( G \) that normalizes \( H \), the size of \( G \) is of the form \( p^\alpha \nu(pm + 1) \).

Sylow did not have the abstract concept of a group: all groups for him arose as subgroups of symmetric groups, so groups were always “groupes de substitutions.” The condition that an element \( x \in G \) is “permutable” with a subgroup \( H \) means \( xH = Hx \), or in other words \( x \in N(H) \). The end of the first part of his theorem says the normalizer of a Sylow subgroup has index \( pm + 1 \) for some \( m \), which means the index is \( \equiv 1 \mod p \).

4. Analogues of the Sylow Theorems

There are analogues of the first two Sylow theorems and Theorem 2.1 for other types of subgroups.

1. A Hall subgroup of a finite group \( G \) is a subgroup \( H \) whose order and index are relatively prime. For example, in a group of order 60 a subgroup of order 12 has index 5 and thus is a Hall subgroup. A \( p \)-subgroup is a Hall subgroup if and only if it is a Sylow subgroup. In 1928 Philip Hall proved in every solvable group of order \( n \) that there is a Hall subgroup of each order \( d \) dividing \( n \) where \( (d, n/d) = 1 \) and two Hall subgroups with the same order are conjugate. Also the normalizer of a Hall subgroup of a solvable group is its own normalizer. Conversely, Hall proved that a finite group of order \( n \) containing a Hall subgroup of order \( d \) for each \( d \) dividing \( n \) such that \( (d, n/d) = 1 \) has to be a solvable group.

2. In a compact connected Lie group \( G \), maximal tori (maximal connected abelian subgroups of \( G \)) satisfy properties analogous to Sylow subgroups: they exist, every torus is in a maximal torus, and all maximal tori are conjugate. The proof of conjugacy uses the Lefschetz fixed point theorem. Like normalizers of Sylow subgroups,

\[1\]We modify some of his notation: he wrote the subgroup as \( g \), not \( H \), and the prime as \( n \), not \( p \).
the normalizer of a maximal torus is its own normalizer. Unlike Sylow subgroups, maximal tori are always abelian and every element of $G$ is in some maximal torus.

(3) In a connected linear algebraic group, maximal connected unipotent subgroups are like Sylow subgroups: they exist, every connected unipotent subgroup is in a maximal connected unipotent subgroup, and all maximal connected unipotent subgroups are conjugate. The proof of conjugacy uses the Borel fixed point theorem. The normalizer of a maximal connected unipotent subgroup is called a Borel subgroup, and like normalizers of Sylow subgroups each Borel subgroup is its own normalizer.

References