THE SYLOW THEOREMS

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1. Introduction

The converse of Lagrange’s theorem is false: if $G$ is a finite group and $d \mid |G|$, then there may not be a subgroup of $G$ with order $d$. The simplest example of this is the group $A_4$, of order 12, which has no subgroup of order 6. The Norwegian mathematician Peter Ludwig Sylow [1] discovered that a converse result is true when $d$ is a prime power: if $p$ is a prime number and $p^k \mid |G|$ then $G$ must contain a subgroup of order $p^k$. Sylow also discovered important relations among the subgroups with order the largest power of $p$ dividing $|G|$, such as the fact that all subgroups of that order are conjugate to each other.

For example, a group of order 100 = $2^2 \cdot 5^2$ must contain subgroups of order 1, 2, 4, 5, and 25, the subgroups of order 4 are conjugate to each other, and the subgroups of order 25 are conjugate to each other. It is not necessarily the case that the subgroups of order 2 are conjugate or that the subgroups of order 5 are conjugate.

Definition 1.1. Let $G$ be a finite group and $p$ be a prime. A subgroup of $G$ whose order is the highest power of $p$ dividing $|G|$ is called a $p$-Sylow subgroup of $G$. A $p$-Sylow subgroup for some $p$ is called a Sylow subgroup.

In a group of order 100, a 2-Sylow subgroup has order 4, a 5-Sylow subgroup has order 25, and a $p$-Sylow subgroup is trivial if $p \neq 2$ or 5.

In a group of order 12, a 2-Sylow subgroup has order 4, a 3-Sylow subgroup has order 3, and a $p$-Sylow subgroup is trivial if $p > 3$. Let’s look at a few examples of Sylow subgroups in groups of order 12.

Example 1.2. In $\mathbb{Z}/(12)$, the only 2-Sylow subgroup is $\{0,3,6,9\} = \langle 3 \rangle$ and the only 3-Sylow subgroup is $\{0,4,8\} = \langle 4 \rangle$.

Example 1.3. In $A_4$ there is one subgroup of order 4, so the only 2-Sylow subgroup is

$$\{(1), (12)(34), (13)(24), (14)(23)\} = \langle (12)(34), (14)(23) \rangle.$$ 

There are four 3-Sylow subgroups:

$$\{(1), (123), (132)\} = \langle (123) \rangle, \quad \{(1), (124), (142)\} = \langle (124) \rangle,$$

$$\{(1), (134), (143)\} = \langle (134) \rangle, \quad \{(1), (234), (243)\} = \langle (234) \rangle.$$ 

Example 1.4. In $D_6$ there are three 2-Sylow subgroups:

$$\{1, r^3, s, r^3s\} = \langle r^3, s \rangle, \quad \{1, r^3, rs, r^4s\} = \langle r^3, rs \rangle, \quad \{1, r^3, r^2s, r^5s\} = \langle r^3, r^2s \rangle.$$ 

The only 3-Sylow subgroup of $D_6$ is $\{1, r^2, r^4\} = \langle r^2 \rangle$.

In a group of order 24, a 2-Sylow subgroup has order 8 and a 3-Sylow subgroup has order 3. Let’s look at two examples.
Example 1.5. In $S_4$, the 3-Sylow subgroups are the 3-Sylow subgroups of $A_4$ (an element of 3-power order in $S_4$ must be a 3-cycle, and they all lie in $A_4$). We determined the 3-Sylow subgroups of $A_4$ in Example 1.3; there are four of them.

There are three 2-Sylow subgroups of $S_4$, and they are interesting to work out since they can be understood as copies of $D_4$ inside $S_4$. The number of ways to label the four vertices of a square as 1, 2, 3, and 4 is $4! = 24$, but up to rotations and reflections of the square there are really just three different ways of carrying out the labeling, as follows.

\[
\begin{array}{c|c}
1 & 2 \\
3 & 4 \\
\end{array} 
\quad \begin{array}{c|c}
2 & 3 \\
1 & 4 \\
\end{array} 
\quad \begin{array}{c|c}
3 & 2 \\
1 & 4 \\
\end{array}
\]

Every other labeling of the square is a rotated or reflected version of one of these three squares. For example, the square below is obtained from the middle square above by reflecting across a horizontal line through the middle of the square.

\[
\begin{array}{c|c}
1 & 3 \\
2 & 4 \\
\end{array}
\]

When $D_4$ acts on a square with labeled vertices, each motion of $D_4$ creates a permutation of the four vertices, and this permutation is an element of $S_4$. For example, a 90 degree rotation of the square is a 4-cycle on the vertices. In this way we obtain a copy of $D_4$ inside $S_4$. The three essentially different labelings of the vertices of the square above embed $D_4$ into $S_4$ as three different subgroups of order 8:

\[
\{1, (1234), (1432), (12)(34), (13)(24), (14)(23), (13), (24)\} = \langle (1234), (13) \rangle,
\]

\[
\{1, (1243), (1342), (12)(34), (14)(23), (12), (13)(24), (14), (23)\} = \langle (1243), (14) \rangle,
\]

\[
\{1, (1324), (1423), (12)(34), (12)(24), (13), (24), (14), (23)\} = \langle (1324), (12) \rangle.
\]

These are the 2-Sylow subgroups of $S_4$.

Example 1.6. The group $\text{SL}_2(\mathbb{Z}/(3))$ has order 24. An explicit tabulation of the elements of this group reveals that there are only 8 elements in the group with 2-power order:

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},
\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.
\]

These form the only 2-Sylow subgroup, which is isomorphic to $Q_8$ by labeling the matrices in the first row as $1, i, j, k$ and the matrices in the second row as $-1, -i, -j, -k$.

There are four 3-Sylow subgroups: $\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$, $\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rangle$, $\langle \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \rangle$, and $\langle \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \rangle$. 
Here are the Sylow theorems. They are often given in three parts. The result we call Sylow III* is not always stated explicitly as part of the Sylow theorems.

**Theorem 1.7 (Sylow I).** A finite group $G$ has a $p$-Sylow subgroup for every prime $p$ and each $p$-subgroup of $G$ lies in some $p$-Sylow subgroup of $G$.

**Theorem 1.8 (Sylow II).** For each prime $p$, the $p$-Sylow subgroups of $G$ are conjugate.

**Theorem 1.9 (Sylow III).** For each prime $p$, let $n_p$ be the number of $p$-Sylow subgroups of $G$. Write $|G| = p^k m$, where $p$ doesn’t divide $m$. Then

\[ n_p \equiv 1 \mod p \quad \text{and} \quad n_p \mid m. \]

**Theorem 1.10 (Sylow III*).** For each prime $p$, let $n_p$ be the number of $p$-Sylow subgroups of $G$. Then $n_p = [G : N(P)]$, where $P$ is a $p$-Sylow subgroup and $N(P)$ is its normalizer.

The existence part of Sylow I has been illustrated in all the previous examples.

Sylow II says for two $p$-Sylow subgroups $H$ and $K$ of $G$ that there is some $g \in G$ such that $gHg^{-1} = K$. This is illustrated in the table below, where Example 1.2 is skipped since $\mathbb{Z}/(12)$ is abelian.

<table>
<thead>
<tr>
<th>Example</th>
<th>Group</th>
<th>Size</th>
<th>$p$</th>
<th>$H$</th>
<th>$K$</th>
<th>$g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3</td>
<td>$A_4$</td>
<td>12</td>
<td>3</td>
<td>\langle (123) \rangle</td>
<td>\langle (124) \rangle</td>
<td>(243)</td>
</tr>
<tr>
<td>1.4</td>
<td>$D_6$</td>
<td>12</td>
<td>2</td>
<td>\langle r^3, s \rangle</td>
<td>\langle r^3, rs \rangle</td>
<td>r^2</td>
</tr>
<tr>
<td>1.5</td>
<td>$S_4$</td>
<td>24</td>
<td>2</td>
<td>\langle (1234), (13) \rangle</td>
<td>\langle (1243), (14) \rangle</td>
<td>(34)</td>
</tr>
<tr>
<td>1.6</td>
<td>$\text{SL}_2(\mathbb{Z}/(3))$</td>
<td>24</td>
<td>3</td>
<td>\langle \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix} \rangle</td>
<td>\langle \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix} \rangle</td>
<td>\begin{pmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{pmatrix}</td>
</tr>
</tbody>
</table>

When trying to conjugate one cyclic subgroup to another cyclic subgroup, beware: not all generators of the two groups have to be conjugate. For example, in $A_4$ the subgroups $\langle (123) \rangle = \{ (1), (123), (132) \}$ and $\langle (124) \rangle = \{ (1), (124), (142) \}$ are conjugate, but the conjugacy class of $(123)$ in $A_4$ is $\{ (123), (142), (134), (243) \}$, so there’s no way to conjugate $(123)$ to $(124)$ by an element of $A_4$; we must conjugate $(123)$ to $(142)$. The 3-cycles $(123)$ and $(124)$ are conjugate in $S_4$, but not in $A_4$. Similarly, $\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$ and $\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle$ are conjugate in $\text{GL}_2(\mathbb{Z}/(3))$ but not in $\text{SL}_2(\mathbb{Z}/(3))$, so when Sylow II says the subgroups $\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$ and $\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle$ are conjugate in $\text{SL}_2(\mathbb{Z}/(3))$ a conjugating matrix must send $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Let’s see what Sylow III tells us about the number of 2-Sylow and 3-Sylow subgroups of a group of order 12. For $p = 2$ and $p = 3$ in Sylow III, the divisibility conditions are $n_2 \mid 3$ and $n_3 \mid 4$ and the congruence conditions are $n_2 \equiv 1 \mod 2$ and $n_3 \equiv 1 \mod 3$. The divisibility conditions imply $n_2$ is 1 or 2 and $n_3$ is 1, 2, or 4. The congruence $n_2 \equiv 1 \mod 2$ tells us nothing new (1 and 3 are both odd), but the congruence $n_3 \equiv 1 \mod 3$ rules out the option $n_3 = 2$. Therefore $n_2$ is 1 or 3 and $n_3$ is 1 or 4 when $|G| = 12$.

If $|G| = 24$ we again find $n_2$ is 1 or 3 while $n_3$ is 1 or 4. (For instance, from $n_3 \mid 8$ and $n_3 \equiv 1 \mod 3$ the only choices are $n_3 = 1$ and $n_3 = 4$.) Therefore as soon as we find more than one 2-Sylow subgroup there must be three of them, and as soon as we find more than one 3-Sylow subgroup there must be four of them. The table below shows the values of $n_2$ and $n_3$ in the examples above.

<table>
<thead>
<tr>
<th>Example</th>
<th>Group</th>
<th>Size</th>
<th>$n_2$</th>
<th>$n_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>$\mathbb{Z}/(12)$</td>
<td>12</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1.3</td>
<td>$A_4$</td>
<td>12</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>1.4</td>
<td>$D_6$</td>
<td>12</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>1.5</td>
<td>$S_4$</td>
<td>24</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1.6</td>
<td>$\text{SL}_2(\mathbb{Z}/(3))$</td>
<td>24</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>
2. Proof of the Sylow Theorems

Our proof of the Sylow theorems will use group actions, which we assume the reader knows. The table below is a summary. For each theorem the table lists a group, a set it acts on, and the action. Let Syl$_p(G)$ be the set of $p$-Sylow subgroups of $G$, so $n_p = |\text{Syl}_p(G)|$.

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Group</th>
<th>Set</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sylow I</td>
<td>$p$-subgroup $H$</td>
<td>$G/H$</td>
<td>left mult.</td>
</tr>
<tr>
<td>Sylow II</td>
<td>$p$-Sylow subgroup $Q$</td>
<td>$G/P$</td>
<td>left mult.</td>
</tr>
<tr>
<td>Sylow III ($n_p \equiv 1 \mod p$)</td>
<td>$P \in \text{Syl}_p(G)$</td>
<td>$\text{Syl}_p(G)$</td>
<td>conjugation</td>
</tr>
<tr>
<td>Sylow III ($n_p</td>
<td>m$)</td>
<td>$G$</td>
<td>$\text{Syl}_p(G)$</td>
</tr>
<tr>
<td>Sylow III*</td>
<td>$G$</td>
<td>$\text{Syl}_p(G)$</td>
<td>conjugation</td>
</tr>
</tbody>
</table>

The two conclusions of Sylow III are listed separately in the table since they are proved using different group actions.

Our proofs will usually involve the action of a $p$-group on a set and use the fixed-point congruence for such actions: $|X| \equiv |\text{Fix}_\Gamma(X)| \mod p$, where $X$ is a finite set being acted on by a finite $p$-group $\Gamma$ and $\text{Fix}_\Gamma(X)$ is the fixed points of $\Gamma$ in $X$.

**Proof of Sylow I**: Let $p^k$ be the highest power of $p$ in $|G|$. The result is obvious if $k = 0$, since the trivial subgroup is a $p$-Sylow subgroup, so we can take $k \geq 1$, hence $p | |G|$.

Our strategy for proving Sylow I is to **prove a stronger result**: there is a subgroup of order $p^i$ for $0 \leq i \leq k$. More specifically, if $|H| = p^i$ and $i < k$, we will show there is a $p$-subgroup $H' \supset H$ with $[H' : H] = p$, so $|H'| = p^{i+1}$. Then, starting with $H$ as the trivial subgroup, we can repeat this process with $H'$ in place of $H$ to create increasingly larger subgroups

\[
\{e\} = H_0 \subset H_1 \subset H_2 \subset \cdots
\]

where $|H_i| = p^i$, and after $k$ steps we reach $H_k$, which is a $p$-Sylow subgroup of $G$. And if we start with $H$ as a $p$-subgroup, we will have shown $H$ is contained in a $p$-Sylow subgroup.

Consider the left multiplication action of $H$ on the left cosets $G/H$ (this need not be a group). This is an action of a finite $p$-group $H$ on the set $G/H$, so by the fixed-point congruence for actions of nontrivial $p$-groups,

\[(2.1) \quad |G/H| \equiv |\text{Fix}_H(G/H)| \mod p.
\]

Here is what it means for $gH$ in $G/H$ to be fixed by the group $H$ acting by left multiplication:

\[
hgH = gH \quad \text{for all} \quad h \in H \quad \iff \quad hg \in gH \quad \text{for all} \quad h \in H
\]

\[
\iff \quad g^{-1}hg \in H \quad \text{for all} \quad h \in H
\]

\[
\iff \quad g^{-1}Hg \subseteq H
\]

\[
\iff \quad g^{-1}Hg = H \quad \text{because} \quad |g^{-1}Hg| = |H|
\]

\[
\iff \quad g \in N(H).
\]

Thus $\text{Fix}_H(G/H) = \{gH : g \in N(H)\} = N(H)/H$, so (2.1) becomes

\[(2.2) \quad |G : H| \equiv |N(H) : H| \mod p.
\]

Because $H \triangleleft N(H)$, $N(H)/H$ is a group.

When $|H| = p^i$ and $i < k$, the index $[G : H]$ is divisible by $p$, so the congruence (2.2) implies $|N(H) : H|$ is divisible by $p$, so $N(H)/H$ is a group with order divisible by $p$. Thus $N(H)/H$ has a subgroup of order $p$ by Cauchy’s theorem. All subgroups of the quotient group $N(H)/H$ have the form $H'/H$, where $H'$ is a subgroup between $H$ and
N(H). Therefore a subgroup of order p in N(H)/H is H'/H such that [H':H] = p, so |H'| = p|H| = p^{j+1}. This can be repeated until we reach a subgroup of order p^k, and we’re done.

**Proof of Sylow II:** For p-Sylow subgroups P and Q, we want to show they are conjugate.

Consider the action of Q on G/P by left multiplication. Since Q is a finite p-group,

$$|G/P| ≡ |\text{Fix}_Q(G/P)| \mod p.$$ 

The left side is $[G:P]$, which is nonzero modulo p since P is a p-Sylow subgroup. Thus $|\text{Fix}_Q(G/P)|$ can’t be 0, so there is a fixed point in G/P. Call it gP. That is, qgP = gP for all q ∈ Q. Equivalently, qg ∈ gP for all q ∈ Q, so Q ⊂ gPg^{-1}. Therefore $Q = gPg^{-1}$, since Q and gPg^{-1} have the same size and we’re done.

**Proof of Sylow III:** We will prove $n_p ≡ 1 \mod p$ and then $n_p | m$.

To show $n_p ≡ 1 \mod p$, consider the action of P on the set Syl_p(G) by conjugation. The size of Syl_p(G) is $n_p$. Since P is a finite p-group, by the fixed-point congruence we have

$$n_p ≡ |\{\text{fixed points}\}| \mod p.$$ 

Fixed points for P acting by conjugation on Syl_p(G) are Q ∈ Syl_p(G) such that gQg^{-1} = Q for all g ∈ P. One choice for Q is P. For all such Q, P ⊂ N(Q). Also Q ⊂ N(Q), so P and Q are p-Sylow subgroups in N(Q). Applying Sylow II to the group N(Q), P and Q are conjugate in N(Q). Since Q < N(Q), the only subgroup of N(Q) conjugate to Q is Q, so P = Q. Thus P is the only fixed point when P acts on Syl_p(G), so $n_p ≡ 1 \mod p$.

To show $n_p | m$, consider the action of G by conjugation on Syl_p(G). Since the p-Sylow subgroups are conjugate to each other (Sylow II), there is one orbit. A set on which a group acts with one orbit has size dividing the size of the group, so $n_p | |G|$. From $n_p ≡ 1 \mod p$, the number $n_p$ is relatively prime to p, so $n_p | m$ and we’re done.

**Proof of Sylow III':** Let P be a p-Sylow subgroup of G and let G act on Syl_p(G) by conjugation. By the orbit-stabilizer formula,

$$n_p = |\text{Syl}_p(G)| = [G:\text{Stab}_P].$$ 

The stabilizer Stab_P of the “point” P in Syl_p(G) (viewing P as a point is why we write {P}) is

$$\text{Stab}_P = \{g : gPg^{-1} = P\} = N(P).$$ 

Thus $n_p = |G:N(P)|$ and we’re done.

In the proof of Sylow I, we saw that if H is a p-subgroup of G that is not a p-Sylow subgroup then N(H) is strictly larger than H. What can be said about N(P) when P is a p-Sylow subgroup? It may or may not be larger than P, but we will show that taking the normalizer a second time will not give anything new.

**Theorem 2.1.** Let P be a p-Sylow subgroup of a finite group G. Then N(N(P)) = N(P).

More generally, if H is a subgroup of G that contains N(P) then N(H) = H.

**Proof.** We will prove H ⊂ N(H) and N(H) ⊂ H. The containment H ⊂ N(H) is easy.

To prove N(H) ⊂ H let $x ∈ N(H)$, so $xHx^{-1} = H$. Since $P ⊂ N(P) ⊂ H$ we have $xPx^{-1} ⊂ xHx^{-1} = H$, so P and $xPx^{-1}$ are both p-Sylow subgroups of H. By Sylow II for the group H, there is $y ∈ H$ such that $xPx^{-1} = yPy^{-1}$. Thus $y^{-1}xP(y^{-1}x)^{-1} = P$, so $y^{-1}x ∈ N(P) ⊂ H$, so $x ∈ yH = H$. □
3. Historical Remarks

Sylow’s proof of his theorems appeared in [1]. Here is what he showed (of course, without using the label “Sylow subgroup”).

1. There exist \( p \)-Sylow subgroups. Moreover, \([G : N(P)] \equiv 1 \text{ mod } p\) for each \( p \)-Sylow subgroup \( P \).

2. Let \( P \) be a \( p \)-Sylow subgroup. The number of \( p \)-Sylow subgroups is \([G : N(P)]\). All \( p \)-Sylow subgroups are conjugate.

3. Each finite \( p \)-group \( G \) with size \( p^k \) contains an increasing chain of subgroups

\[ \{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_k \subset G, \]

where each subgroup has index \( p \) in the next one. In particular, \(|G_i| = p^i \) for all \( i \).

Here is how Sylow phrased his first theorem (the first item on the above list):\(^1\)

Si \( p^{\alpha} \) désigne la plus grande puissance du nombre premier \( p \) qui divise l’ordre du groupe \( G \), ce groupe contient un autre \( H \) de l’ordre \( p^{\alpha} \); si de plus \( p^{\alpha} \nu \) désigne l’ordre du plus grand groupe contenu dans \( G \) dont les substitutions sont permutables à \( H \), l’ordre de \( G \) sera de la forme \( p^{\alpha} \nu(p^m + 1) \).

In English, using current terminology, this says

If \( p^{\alpha} \) is the largest power of the prime \( p \) which divides the size of the group \( G \), this group contains a subgroup \( H \) of order \( p^{\alpha} \); if moreover \( p^{\alpha} \nu \) is the size of the largest subgroup of \( G \) that normalizes \( H \), the size of \( G \) is of the form \( p^{\alpha} \nu(p^m + 1) \).

Sylow did not have the abstract concept of a group: all groups for him arose as subgroups of symmetric groups, so groups were always “groupes de substitutions.” The condition that an element \( x \in G \) is “permutable” with a subgroup \( H \) means \( xH = Hx \), or in other words \( x \in N(H) \). The end of the first part of his theorem says the normalizer of a Sylow subgroup has index \( pm + 1 \) for some \( m \), which means the index is \( \equiv 1 \text{ mod } p \).

4. Analogues of the Sylow Theorems

There are analogues of the first two Sylow theorems and Theorem 2.1 for other types of subgroups.

1. A Hall subgroup of a finite group \( G \) is a subgroup \( H \) whose order and index are relatively prime. For example, in a group of order 60 a subgroup of order 12 has index 5 and thus is a Hall subgroup. A \( p \)-subgroup is a Hall subgroup if and only if it is a Sylow subgroup. In 1928 Philip Hall proved in every solvable group of order \( n \) that there is a Hall subgroup of each order \( d \) dividing \( n \) where \((d, n/d) = 1 \) and two Hall subgroups with the same order are conjugate. Also the normalizer of a Hall subgroup of a solvable group is its own normalizer. Conversely, Hall proved that a finite group of order \( n \) containing a Hall subgroup of order \( d \) for each \( d \) dividing \( n \) such that \((d, n/d) = 1 \) has to be a solvable group.

2. In a compact connected Lie group \( G \), maximal tori (maximal connected abelian subgroups of \( G \)) satisfy properties analogous to Sylow subgroups: they exist, every torus is in a maximal torus, and all maximal tori are conjugate. The proof of conjugacy uses the Lefschetz fixed point theorem. Like normalizers of Sylow subgroups,

\(^1\)We modify some of his notation: he wrote the subgroup as \( g \), not \( H \), and the prime as \( n \), not \( p \).
the normalizer of a maximal torus is its own normalizer. Unlike Sylow subgroups, maximal tori are always abelian and every element of $G$ is in some maximal torus.

(3) In a connected linear algebraic group, maximal connected unipotent subgroups are like Sylow subgroups: they exist, every connected unipotent subgroup is in a maximal connected unipotent subgroup, and all maximal connected unipotent subgroups are conjugate. The proof of conjugacy uses the Borel fixed point theorem. The normalizer of a maximal connected unipotent subgroup is called a Borel subgroup, and like normalizers of Sylow subgroups each Borel subgroup is its own normalizer.

References