

GROUPS OF ORDER p^3

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For each prime p , we want to describe the groups of order p^3 up to isomorphism. This was done for $p = 2$ by Cayley [3, 4] in 1859 and 1889 and Kempe [8, pp. 38–39, 45] in 1886, and for odd p by Cole and Glover [5, pp. 196–201], Hölder [7, pp. 371–373] and Young [13, pp. 133–139] independently in 1893. The groups were described by them using generators and relations, which sometimes does not lead to convincing arguments that the groups constructed to be of order p^3 really exist.¹

From the cyclic decomposition of finite abelian groups, there are three abelian groups of order p^3 up to isomorphism: $\mathbf{Z}/(p^3)$, $\mathbf{Z}/(p^2) \times \mathbf{Z}/(p)$, and $\mathbf{Z}/(p) \times \mathbf{Z}/(p) \times \mathbf{Z}/(p)$. These are nonisomorphic since they have different maximal orders for their elements: p^3 , p^2 , and p respectively. We will show there are two nonabelian groups of order p^3 up to isomorphism. The descriptions of these two groups will be different for $p = 2$ and $p \neq 2$, so we will treat these cases separately after the following lemma.

Lemma 1. *Let p be prime and G be a nonabelian group of order p^3 with center Z . Then $|Z| = p$, $G/Z \cong (\mathbf{Z}/(p)) \times (\mathbf{Z}/(p))$, and $[G, G] = Z$.*

Proof. Since G is a nontrivial group of p -power order, its center is nontrivial. Therefore $|Z| = p, p^2$, or p^3 . Since G is nonabelian, $|Z| \neq p^3$. For a group G , if G/Z is cyclic then G is abelian. So G being nonabelian forces G/Z to be noncyclic. Therefore $|G/Z| \neq p$, so $|Z| \neq p^2$. The only choice left is $|Z| = p$, so G/Z has order p^2 .

Up to isomorphism the only groups of order p^2 are $\mathbf{Z}/(p^2)$ and $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$. Since G/Z is noncyclic, $G/Z \cong \mathbf{Z}/(p) \times \mathbf{Z}/(p)$.

Since G/Z is abelian, we have $[G, G] \subset Z$. Because $|Z| = p$ and $[G, G]$ is nontrivial, necessarily $[G, G] = Z$. □

Theorem 2. *A nonabelian group of order 8 is isomorphic to D_4 or to Q_8 .*

The groups D_4 and Q_8 are not isomorphic since there are 5 elements of order 2 in D_4 and only one element of order 2 in Q_8 .

Proof. Let G be nonabelian of order 8. The nonidentity elements in G have order 2 or 4. If $g^2 = 1$ for all $g \in G$ then G is abelian, so some $x \in G$ must have order 4.

Let $y \in G - \langle x \rangle$. The subgroup $\langle x, y \rangle$ properly contains $\langle x \rangle$, so $\langle x, y \rangle = G$. Since G is nonabelian, x and y do not commute.

Since $\langle x \rangle$ has index 2 in G , it is a normal subgroup. Therefore $xyx^{-1} \in \langle x \rangle$:

$$xyx^{-1} \in \{1, x, x^2, x^3\}.$$

Since xyx^{-1} has order 4, $xyx^{-1} = x$ or $xyx^{-1} = x^3 = x^{-1}$. The first option is not possible, since it says x and y commute, which they don't. Therefore

$$xyx^{-1} = x^{-1}.$$

¹The page <https://math.stackexchange.com/questions/1023341> gives a nonobvious description of the trivial group by generators and relations.

The group $G/\langle x \rangle$ has order 2, so $y^2 \in \langle x \rangle$:

$$y^2 \in \{1, x, x^2, x^3\}.$$

Since y has order 2 or 4, y^2 has order 1 or 2. Thus $y^2 = 1$ or $y^2 = x^2$.

Putting this together, $G = \langle x, y \rangle$ where either

$$x^4 = 1, \quad y^2 = 1, \quad yxy^{-1} = x^{-1}$$

or

$$x^4 = 1, \quad y^2 = x^2, \quad yxy^{-1} = x^{-1}.$$

In the first case $G \cong D_4$ and in the second case $G \cong Q_8$. \square

From now on we take $p \neq 2$. The two nonabelian groups of order p^3 , up to isomorphism, will turn out to be

$$\text{Heis}(\mathbf{Z}/(p)) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbf{Z}/(p) \right\}$$

and

$$G_p = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbf{Z}/(p^2), a \equiv 1 \pmod{p} \right\} = \left\{ \begin{pmatrix} 1 + pm & b \\ 0 & 1 \end{pmatrix} : m, b \in \mathbf{Z}/(p^2) \right\},$$

where m actually only matters modulo p .² These two constructions both make sense at the prime 2, but in that case the two groups are isomorphic to each other, as we'll see below.

We can distinguish $\text{Heis}(\mathbf{Z}/(p))$ from G_p for $p \neq 2$ by counting elements of order p . In $\text{Heis}(\mathbf{Z}/(p))$,

$$(1) \quad \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & na & nb + \frac{n(n-1)}{2}ac \\ 0 & 1 & nc \\ 0 & 0 & 1 \end{pmatrix}$$

for $n \in \mathbf{Z}$, so

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^p = \begin{pmatrix} 1 & 0 & \frac{p(p-1)}{2}ac \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

When $p \neq 2$, $\frac{p(p-1)}{2} \equiv 0 \pmod{p}$, so all nonidentity elements of $\text{Heis}(\mathbf{Z}/(p))$ have order p . On the other hand, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G_p$ has order p^2 since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. So $G_p \not\cong \text{Heis}(\mathbf{Z}/(p))$.

At the prime 2, $\text{Heis}(\mathbf{Z}/(2))$ and G_2 each contain more than one element of order 2, so $\text{Heis}(\mathbf{Z}/(2))$ and G_2 are both isomorphic to D_4 .

Let's look at how matrices combine and decompose in $\text{Heis}(\mathbf{Z}/(p))$ and G_p when $p \neq 2$, since this will inform some of our computations later in an abstract nonabelian group of order p^3 . In $\text{Heis}(\mathbf{Z}/(p))$,

$$(2) \quad \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a + a' & b + b' + ac' \\ 0 & 1 & c + c' \\ 0 & 0 & 1 \end{pmatrix}$$

and in G_p

$$(3) \quad \begin{pmatrix} 1 + pm & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + pm' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + p(m + m') & b + b' + pmb' \\ 0 & 1 \end{pmatrix}.$$

²The notation G_p for this group is not standard. I don't know a standard "matrix group" notation for it.

In $\text{Heis}(\mathbf{Z}/(p))$,

$$\begin{aligned} \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^c \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^a \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^b \quad \text{by (1)} \end{aligned}$$

and a particular commutator is

$$\left[\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So if we set

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

then

$$(4) \quad \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = y^c x^a [x, y]^b.$$

In $G_p \subset \text{Aff}(\mathbf{Z}/(p^2))$,

$$\begin{pmatrix} 1+pm & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+pm & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^b \begin{pmatrix} 1+p & 0 \\ 0 & 1 \end{pmatrix}^m.$$

If we set

$$x = \begin{pmatrix} 1+p & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then

$$\begin{pmatrix} 1+pm & b \\ 0 & 1 \end{pmatrix} = y^b x^m$$

and

$$[x, y] = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} = y^p.$$

Lemma 3. In a group G , if g and h commute with $[g, h]$ then $[g^m, h^n] = [g, h]^{mn}$ for all m and n in \mathbf{Z} , and $g^n h^n = (gh)^n [g, h]^{\binom{n}{2}}$.

Proof. Exercise. □

Theorem 4. For primes $p \neq 2$, a nonabelian group of order p^3 is isomorphic to $\text{Heis}(\mathbf{Z}/(p))$ or G_p .

Proof. Let G be a nonabelian group of order p^3 . Each $g \neq 1$ in G has order p or p^2 .

By Lemma 1, we can write $G/Z = \langle \bar{x}, \bar{y} \rangle$ and $Z = \langle z \rangle$. For $g \in G$, $g \equiv x^i y^j \pmod{Z}$ for some integers i and j , so $g = x^i y^j z^k = z^k x^i y^j$ for some $k \in \mathbf{Z}$. If x and y commute then G is abelian (since z^k commutes with x and y), which is a contradiction. Thus x and y do not commute. Therefore $[x, y] = xyx^{-1}y^{-1} \in Z$ is nontrivial, so $Z = \langle [x, y] \rangle$. Therefore we can use $[x, y]$ for z , showing $G = \langle x, y \rangle$.

Let's see what the product of two elements of G looks like. Using Lemma 3,

$$(5) \quad x^i y^j = y^j x^i [x, y]^{ij}, \quad y^j x^i = x^i y^j [x, y]^{-ij}.$$

This shows we can move every power of y past every power of x on either side, at the cost of introducing a (commuting) power of $[x, y]$. So every element of $G = \langle x, y \rangle$ has the form $y^j x^i [x, y]^k$. (We write in this order because of (4).) A product of two such terms is

$$\begin{aligned} y^c x^a [x, y]^b \cdot y^{c'} x^{a'} [x, y]^{b'} &= y^c (x^a y^{c'}) x^{a'} [x, y]^{b+b'} \\ &= y^c (y^{c'} x^a [x, y]^{ac'}) x^{a'} [x, y]^{b+b'} \quad \text{by (5)} \\ &= y^{c+c'} x^{a+a'} [x, y]^{b+b'+ac'}. \end{aligned}$$

Here the exponents are all integers. Comparing this with (2), it appears we have a homomorphism $\text{Heis}(\mathbf{Z}/(p)) \rightarrow G$ by

$$(6) \quad \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto y^c x^a [x, y]^b.$$

After all, we just showed multiplication of such triples $y^c x^a [x, y]^b$ behaves like multiplication in $\text{Heis}(\mathbf{Z}/(p))$. But there is a catch: the matrix entries a , b , and c in $\text{Heis}(\mathbf{Z}/(p))$ are integers modulo p , so the “function” (6) from $\text{Heis}(\mathbf{Z}/(p))$ to G is only well-defined if x , y , and $[x, y]$ all have p -th power 1 (so exponents on them only matter mod p). Since $[x, y]$ is in the center of G , a subgroup of order p , its exponents only matter modulo p . But maybe x or y could have order p^2 .

Well, if x and y both have order p , then there is no problem with (6). It is a well-defined function $\text{Heis}(\mathbf{Z}/(p)) \rightarrow G$ that is a homomorphism. Since its image contains x and y , the image contains $\langle x, y \rangle = G$, so the function is onto. Both $\text{Heis}(\mathbf{Z}/(p))$ and G have order p^3 , so our surjective homomorphism is an isomorphism: $G \cong \text{Heis}(\mathbf{Z}/(p))$.

What happens if x or y has order p^2 ? In this case we anticipate that $G \cong G_p$. In G_p , two generators are $g = \begin{pmatrix} 1+p & 0 \\ 0 & 1 \end{pmatrix}$ and $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, where g has order p , h has order p^2 , and $[g, h] = h^p$. We want to show our abstract G also has a pair of generators like this.

Starting with $G = \langle x, y \rangle$ where x or y has order p^2 , without loss of generality let y have order p^2 . It may or may not be the case that x has order p . To show we can change generators to make x have order p , we will look at the p -th power function on G . For all $g \in G$, $g^p \in Z$ since $G/Z \cong \mathbf{Z}/(p) \times \mathbf{Z}/(p)$. Moreover, the p -th power function on G is a *homomorphism*: by Lemma 3, $(gh)^p = g^p h^p [g, h]^{p(p-1)/2}$ and $[g, h]^p = 1$ since $[G, G] = Z$ has order p , so

$$(gh)^p = g^p h^p.$$

Since y^p has order p and $y^p \in Z$, $Z = \langle y^p \rangle$. Therefore $x^p = (y^p)^r$ for some $r \in \mathbf{Z}$, and since the p -th power function on G is a homomorphism we get $(xy^{-r})^p = 1$, with $xy^{-r} \neq 1$ since $x \notin \langle y \rangle$. So xy^{-r} has order p and $G = \langle x, y \rangle = \langle xy^{-r}, y \rangle$. We now rename xy^{-r} as x , so $G = \langle x, y \rangle$ where x has order p and y has order p^2 .

We are not guaranteed that $[x, y] = y^p$, which is one of the relations for the two generators of G_p . How can we force this relation to occur? Well, since $[x, y]$ is a nontrivial element of $[G, G] = Z$, $Z = \langle [x, y] \rangle = \langle y^p \rangle$, so

$$(7) \quad [x, y] = (y^p)^k,$$

where $k \not\equiv 0 \pmod p$. Let ℓ be a multiplicative inverse for $k \pmod p$ and raise both sides of (7) to the ℓ th power: using Lemma 3,

$$[x, y]^\ell = (y^{pk})^\ell \implies [x^\ell, y] = y^p.$$

Since $\ell \not\equiv 0 \pmod p$, $\langle x \rangle = \langle x^\ell \rangle$, so we can rename x^ℓ as x : now $G = \langle x, y \rangle$ where x has order p , y has order p^2 , and $[x, y] = y^p$.

Because $[x, y]$ commutes with x and y and $G = \langle x, y \rangle$, every element of G has the form $y^j x^i [x, y]^k = [x, y]^k y^j x^i = y^{pk+j} x^i$. Let's see how such products multiply:

$$\begin{aligned} y^b x^m \cdot y^{b'} x^{m'} &= y^b (x^m y^{b'}) x^{m'} \\ &= y^b (y^{b'} x^m [x, y]^{mb'}) x^{m'} \\ &= y^{b+b'} x^m (y^p)^{mb'} x^{m'} \\ &= y^{b+b'+pmb'} x^{m+m'}. \end{aligned}$$

Comparing this with (3), we have a homomorphism $G_p \rightarrow G$ by

$$\begin{pmatrix} 1+pm & b \\ 0 & 1 \end{pmatrix} \mapsto y^b x^m.$$

(This function is well-defined since on the left side m matters mod p and b matters mod p^2 while $x^p = 1$ and $y^{p^2} = 1$.) This homomorphism is onto since x and y are in the image, so it is an isomorphism since G_p and G have equal order: $G \cong G_p$. \square

Let's summarize what is known about the count of groups of small p -power order.

- There is one group of order p up to isomorphism.
- There are two groups of order p^2 up to isomorphism: $\mathbf{Z}/(p^2)$ and $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$.
- There are five groups of order p^3 up to isomorphism, but our explicit description of them is not uniform in p since the case $p = 2$ used a separate treatment.

For groups of order p^4 , the count is no longer uniform in p : there are 14 groups of order 2^4 and 15 groups of order p^4 for $p \neq 2$. This is due to Hölder [7] and Young [13]. A recent account of this result by Adler, Garlow, and Wheland is on the arXiv [1]. For groups of order p^5 , the count depends on $p \pmod{12}$ as shown in the table below. This is due to Miller [9] for $p = 2$ and Bagnera [2] for $p > 2$. Tables listing groups of order 32 and 243 are available at Tim Dokchitser's site [6]. The first count of groups of order p^6 is due to Potron [12], with a modern count being made by Newman, O'Brien, and Vaughan-Lee [10]. A count of groups of order p^7 is due to O'Brien and Vaughan-Lee [11].

p	2	3	1 mod 12	5 mod 12	7 mod 12	11 mod 12
Groups of order p^5	51	67	$2p + 71$	$2p + 67$	$2p + 69$	$2p + 65$

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