GROUPS OF ORDER 16

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1. Introduction

There are 14 groups of order 16 up to isomorphism. This was first determined in 1893 by Hölder [2] and Young [5]. A more recent proof of this result was given by Wild [4] in 2005.

These 14 groups are described in Table 1, with the abelian ones listed first. The "GAP" number in the second column is the ordering among the groups of order 16 in the computer algebra package GAP, in case you want to find these groups in GAP or in tables online such as Dockchitser's site [1] where the GAP number is in the rightmost column. The "Wild" number in the third column is the ordering of these groups by Wild [4, Theorem 2].

Group	GAP	Wild	Description
$\mathbf{Z}/(16)$	1	6	Abelian
$\mathbf{Z}/(8) \times \mathbf{Z}/(2)$	5	1	Abelian
$\mathbf{Z}/(4) \times \mathbf{Z}/(4)$	2	13	Abelian
$(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(4)$	10	7	Abelian
$(\mathbf{Z}/(2))^4$	14	0	Abelian
D_8	7	4	
$D_4 \times \mathbf{Z}/(2)$	11	8	
$Q_8 \times \mathbf{Z}/(2)$	12	11	
Q_{16}	9	5	$\langle x, y \mid x^8 = 1, y^4 = 1, x^4 = y^2, yxy^{-1} = x^{-1} \rangle$
$\mathbf{Z}/(8) \rtimes_3 \mathbf{Z}/(2)$	8	2	$(a,b)(c,d) = (a + 3^b c, b + d)$
$\mathbf{Z}/(8) \rtimes_5 \mathbf{Z}/(2)$	6	3	$(a,b)(c,d) = (a + 5^b c, b + d)$
$\mathbf{Z}/(4) \rtimes \mathbf{Z}/(4)$	4	12	$(a,b)(c,d) = (a + (-1)^b c, b + d)$
$(\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(4)$	3	9	$(v,m)(w,n) = (v + A^m w, m + n)$ for $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$
$Q_8 \rtimes \mathbf{Z}/(2)$	13	10	$(g,m)(h,n) = (gi^m h i^{-m}, m + n)$

Table 1. Groups of order 16 up to isomorphism.

The group Q_{16} is a generalized quaternion group.[1] The last five groups in Table 1 use nontrivial semidirect products. The first two come from different nontrivial homomorphisms $\mathbf{Z}/(2) \to (\mathbf{Z}/(8))^\times$ (mapping 1 mod 2 to 3 and 5 mod 8, which are the elements of order 2 in $(\mathbf{Z}/(8))^\times$ other than -1). For the third semidirect product, the only nontrivial homomorphism $\mathbf{Z}/(4) \to \mathrm{Aut}(\mathbf{Z}/(4)) \cong \{\pm 1 \bmod 4\}$ is $b \bmod 4 \mapsto (-1)^b \bmod 4$. For the fourth semidirect product, all nontrivial homomorphisms $\mathbf{Z}/(4) \to \mathrm{Aut}((\mathbf{Z}/(2))^2) = \mathrm{GL}_2(\mathbf{Z}/(2)) \cong S_3$ are conjugate since 1 mod 4 must go to an element of order 2 and all elements of order 2 in $\mathrm{GL}_2(\mathbf{Z}/(2))$ are conjugate. So all nontrivial semidirect products $(\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(4)$ are isomorphic. The final semidirect product makes sense since conjugation on Q_8 by i has order 2 (even if i in Q_8 has order 4, since $i^2 = -1 \in Z(Q_8)$). The group $Q_8 \rtimes \mathbf{Z}/(2)$ is

[1] See Section 3 of https://kconrad.math.uconn.edu/blurbs/grouptheory/genquat.pdf.

unchanged (up to isomorphism) if we define the semidirect product using conjugation by j or k in place of i.

We will not explain here why every group of order 16 is isomorphic to some group in Table 1; for that, see [4]. What we will do, in the next section, is explain why the groups in Table 1 are nonisomorphic. In the course of this task we will see that some nonisomorphic groups of order 16 can have the same number of elements of each order.

2. Distinguishing the groups of order 16

In a group of order 16, every element has order 1, 2, 4, 8, or 16. Table 2 below indicates how many elements have each order in the groups from Table 1.

Group	GAP	Order 1	Order 2	Order 4	Order 8	Order 16
$\mathbf{Z}/(16)$	1	1	1	2	4	8
$\mathbf{Z}/(8) \times \mathbf{Z}/(2)$	5	1	3	4	8	0
$\mathbf{Z}/(4) \times \mathbf{Z}/(4)$	2	1	3	12	0	0
$(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(4)$	10	1	7	8	0	0
$(\mathbf{Z}/(2))^4$	14	1	15	0	0	0
D_8	7	1	9	2	4	0
$D_4 \times \mathbf{Z}/(2)$	11	1	11	4	0	0
$Q_8 \times \mathbf{Z}/(2)$	12	1	3	12	0	0
Q_{16}	9	1	1	10	4	0
$\mathbf{Z}/(8) \rtimes_3 \mathbf{Z}/(2)$	8	1	5	6	4	0
$\mathbf{Z}/(8) \rtimes_5 \mathbf{Z}/(2)$	6	1	3	4	8	0
$\mathbf{Z}/(4) \rtimes \mathbf{Z}/(4)$	4	1	3	12	0	0
$(\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(4)$	3	1	7	8	0	0
$Q_8 \rtimes \mathbf{Z}/(2)$	13	1	7	8	0	0

Table 2. Groups of order 16 with orders of elements.

The counts in Table 2 can let us determine which group in Table 1 some other group of order 16 is isomorphic to.

Example 2.1. Let $G = (\mathbf{Z}/(4) \times \mathbf{Z}/(2)) \rtimes \mathbf{Z}/(2)$, where we make 1 mod 2 in $\mathbf{Z}/(2)$ act on $\mathbf{Z}/(4) \times \mathbf{Z}/(2)$ by negation, which is an automorphism of order 2. Check that G has 11 elements of order 2 (these are the $(v, 0)$ where $v = (2, 0)$, $(2, 1)$, and $(0, 1)$ and all 8 elements $(v, 1)$), so $G \cong D_4 \times \mathbf{Z}/(2)$ by Table 2.

Example 2.2. Let $G = \mathbf{Z}/(8) \rtimes_7 \mathbf{Z}/(2)$, where

$$(a, b)(c, d) = (a + 7^b c, b + d) = (a + (-1)^b c, b + d).$$

Check G is nonabelian with two elements of order 4 (these are $(2, 0)$ and $(6, 0)$), so $G \cong D_8$ by Table 2. In fact, each dihedral group D_n is isomorphic to the semidirect product $\mathbf{Z}/(n) \rtimes \mathbf{Z}/(2)$ where $(a, b)(c, d) = (a + (-1)^b c, b + d)$.

Perhaps surprisingly, some nonisomorphic groups of order 16 have the same number of elements of each order. This is indicated by the coloring of some rows in Table 2:

- $\mathbf{Z}/(8) \times \mathbf{Z}/(2)$ and $\mathbf{Z}/(8) \rtimes_5 \mathbf{Z}/(2)$. These are not isomorphic since the first is abelian and the second isn't. (These groups also have the same subgroup lattice.)

- $\mathbf{Z}/(4) \times \mathbf{Z}/(4)$, $Q_8 \times \mathbf{Z}/(2)$, and $\mathbf{Z}/(4) \rtimes \mathbf{Z}/(4)$. The first one is abelian and the other two are nonabelian. We'll distinguish the nonabelian groups below.
- $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(4)$, $(\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(4)$, and $Q_8 \rtimes \mathbf{Z}/(2)$. The first one is abelian and the other two are nonabelian. We'll distinguish the nonabelian groups below.

That groups of order 16 can be nonisomorphic but have the same number of elements of each order was first pointed out by Miller [3, p. 270] in 1898. This does not happen for groups of order less than 16 or for abelian groups: a finite abelian group is determined up to isomorphism by the number of elements it has of each order. Here is an infinite collection of pairs of nonisomorphic groups with the same number of elements of each order. For odd primes p, the abelian group $(\mathbf{Z}/(p))^3$ and the nonabelian group

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbf{Z}/(p) \right\}$$

have p^3 elements. All non-identity elements in $(\mathbf{Z}/(p))^3$ have order p, and this is true in the matrix group too on account of the power formula

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & na & nb + \frac{n(n-1)}{2}ac \\ 0 & 1 & nc \\ 0 & 0 & 1 \end{pmatrix}$$

for $n \in \mathbf{Z}$, which at $n = p$ becomes

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

because p and $p(p-1)/2$ are 0 in $\mathbf{Z}/(p)$ when $p > 2$. (For $p = 2$, the matrix group has elements of order 4 while all non-identity elements of $(\mathbf{Z}/(2))^3$ have order 2.)

Returning to nonabelian groups in Table 2 with the same number of elements of each order, we can show they are nonisomorphic by looking at their centers. To distinguish between $(\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(4)$ and $Q_8 \rtimes \mathbf{Z}/(2)$, check the center of $(\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(4)$ is $\langle (\binom{0}{0}, 2), (\binom{1}{0}, 0) \rangle \cong (\mathbf{Z}/(2))^2$ and the center of $Q_8 \rtimes \mathbf{Z}/(2)$ is $\langle (i, 1) \rangle \cong \mathbf{Z}/(4)$. Both centers have order 4, with the first being noncyclic and the second being cyclic. Since the centers are not isomorphic, the original groups are not isomorphic.

To distinguish between $Q_8 \times \mathbf{Z}/(2)$ and $\mathbf{Z}/(4) \rtimes \mathbf{Z}/(4)$ we can also use their centers, but in a more subtle way because the centers are isomorphic: $Q_8 \times \mathbf{Z}/(2)$ has center $\{\pm 1\} \times \{0, 1\}$ and $\mathbf{Z}/(4) \rtimes \mathbf{Z}/(4)$ has center $\{0, 2\} \times \{0, 2\}$. Check both centers are noncyclic of order 4, so they are isomorphic. In the center of $Q_8 \times \mathbf{Z}/(2)$, just one nontrivial element is a square in the whole group: $(-1, 0) = (i, 0)^2$. In the center of $\mathbf{Z}/(4) \rtimes \mathbf{Z}/(4)$, $(2, 0)$ and $(0, 2)$ are squares in the whole group: $(2, 0) = (1, 0)^2$ and $(0, 2) = (0, 1)^2$. Even though $Q_8 \times \mathbf{Z}/(2)$, and $\mathbf{Z}/(4) \rtimes \mathbf{Z}/(4)$ have isomorphic centers, the centers don't lie in these groups in the same way (their centers have a different number of squares in the whole group), so the original groups are not isomorphic.

To summarize, we can determine with one exception whether two groups of order 16 are isomorphic by

- counting the number of elements of each order in both groups (or just of order 2 and 4),

- computing the centers of both groups up to isomorphism if the groups have the same number of elements of each order.

Two groups of order 16 that are not distinguished by these two calculations must be isomorphic to $Q_8 \times \mathbf{Z}/(2)$ or $\mathbf{Z}/(4) \rtimes \mathbf{Z}/(4)$, and in that case we can determine if they are isomorphic by checking how many elements in their centers are squares in the whole group.

A group G of order 16 has a nontrivial center Z, and if G is nonabelian then we can show $|Z| \leq 4$ by contradiction. If $|Z| > 4$ then $|Z| = 8$ since G is nonabelian, so $|G/Z| = 2$. A standard result in group theory is that if G/Z is cyclic then G is abelian. If $|G/Z| = 2$ then G/Z is cyclic, so G is abelian and we have a contradiction since G is nonabelian. Thus $|Z| \leq 4$ when $|G| = 16$ and G is nonabelian. Table 3 collects the nine nonabelian groups of order 16 together according to their center (up to isomorphism), which has order 2 or 4.

Center	Groups of order 16 with that center, up to isomorphism
$\mathbf{Z}/(4)$	$\mathbf{Z}/(8) \rtimes_5 \mathbf{Z}/(2)$, $Q_8 \rtimes \mathbf{Z}/(2)$
$(\mathbf{Z}/(2))^2$	$D_4 \times \mathbf{Z}/(2)$, $Q_8 \times \mathbf{Z}/(2)$, $\mathbf{Z}/(4) \rtimes \mathbf{Z}/(4)$, $(\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(4)$
$\mathbf{Z}/(2)$	D_8, Q_{16}, $\mathbf{Z}/(8) \rtimes_3 \mathbf{Z}/(2)$

Table 3. Nonabelian groups of order 16 having a specific center.

References

[1] T. Dockchitser, Group Names, https://people.maths.bris.ac.uk/∼matyd/GroupNames/.

[2] O. Hölder, "Die Gruppen der Ordnungen p^3, pq^2, pqr, p^4," *Math. Ann.* **43** (1893), 301–412. URL https://eudml.org/doc/157685.

[3] G. A. Miller, "The Regular Substitution Groups Whose Orders are Less than 48," *Quart. J. Pure Appl. Math.* **28** (1896), 232–284. URL https://gdz.sub.uni-goettingen.de/id/PPN600494829_0028.

[4] M. Wild "The Groups of Order Sixteen Made Easy" *Amer. Math. Monthly* **112** (2005), 20–31. URL https://www.jstor.org/stable/30037381.

[5] J. Young, "On the determination of the groups whose order is a power of a prime," *Amer. J. Math.* **15** (1893), 124–178. URL https://www.jstor.org/stable/2369564.