The groups of order 12, up to isomorphism, were first determined in the 1880s by Cayley [1] and Kempe [2, pp. 37–43]: Kempe gave a list of 5 groups and Cayley pointed out a few years later that one of Kempe’s groups did not make sense and that Kempe had missed an example, which Cayley provided.

We will use semidirect products to describe the groups of order 12. There turn out to be 5 such groups: 2 are abelian and 3 are nonabelian. The nonabelian groups are an alternating group, a dihedral group, and a third less familiar group.

**Theorem 1.** Every group of order 12 is isomorphic to one of \( \mathbb{Z}/(12) \), \((\mathbb{Z}/(2))^2 \times \mathbb{Z}/(3)\), \(A_4\), \(D_6\), or the nontrivial semidirect product \( \mathbb{Z}/(3) \rtimes \mathbb{Z}/(4) \).

In the proof, we will appeal to an isomorphism property of semidirect products: for each semidirect product \( H \rtimes \varphi K \) and automorphism \( f: K \to K \), \( H \rtimes \varphi K \cong H \rtimes \varphi \circ f K \). This says that precomposing an action of \( K \) on \( H \) by automorphisms (that’s \( \varphi \)) with an automorphism of \( K \) produces an isomorphic semidirect product of \( H \) and \( K \).

**Proof.** Let \(|G| = 12 = 2^2 \cdot 3\). A 2-Sylow subgroup has order 4 and a 3-Sylow subgroup has order 3. We will start by showing \( G \) has a normal 2-Sylow or 3-Sylow subgroup: \( n_2 = 1 \) or \( n_3 = 1 \). From the Sylow theorems,

\[
\begin{align*}
  n_2 &\mid 3, \quad n_2 \equiv 1 \mod 2, \quad n_3 \mid 4, \quad n_3 \equiv 1 \mod 3.
\end{align*}
\]

Therefore \( n_2 = 1 \) or 3 and \( n_3 = 1 \) or 4.

To show \( n_2 = 1 \) or \( n_3 = 1 \), assume \( n_3 \neq 1 \). Then \( n_3 = 4 \). Let’s count elements of order 3. Since each 3-Sylow subgroup has prime order 3, two different 3-Sylow subgroups intersect trivially. Each of the four 3-Sylow subgroups of \( G \) contains two elements of order 3 and they are not in another 3-Sylow subgroup, so the number of elements in \( G \) of order 3 is \( 2n_2 = 8 \). This leaves us with \( 12 - 8 = 4 \) elements in \( G \) not of order 3. A 2-Sylow subgroup has order 4 and contains no elements of order 3, so one 2-Sylow subgroup must account for the remaining 4 elements of \( G \). Thus \( n_2 = 1 \) if \( n_3 \neq 1 \).

Next we show \( G \) is a semidirect product of a 2-Sylow and 3-Sylow subgroup. Let \( P \) be a 2-Sylow subgroup and \( Q \) be a 3-Sylow subgroup of \( G \). Since \( P \) and \( Q \) have relatively prime orders, \( P \cap Q = \{1\} \) and the set \( PQ = \{xy : x \in P, y \in Q\} \) has size \(|P||Q|/|P \cap Q| = 12 = |G|\), so \( G = PQ \). Since \( P \) or \( Q \) is normal in \( G \), \( G \) is a semidirect product of \( P \) and \( Q \): \( G \cong P \rtimes Q \) if \( P \lhd G \) and \( G \cong Q \rtimes P \) if \( Q \lhd G \).

Groups of order 4 are isomorphic to \( \mathbb{Z}/(4) \) or \( (\mathbb{Z}/(2))^2 \), and groups of order 3 are isomorphic to \( \mathbb{Z}/(3) \), so \( G \) is a semidirect product of the form

\[
\begin{align*}
  \mathbb{Z}/(4) \rtimes \mathbb{Z}/(3), \quad (\mathbb{Z}/(2))^2 \rtimes \mathbb{Z}/(3), \quad \mathbb{Z}/(3) \rtimes \mathbb{Z}/(4), \quad \mathbb{Z}/(3) \times (\mathbb{Z}/(2))^2.
\end{align*}
\]

\[\text{1} \] The notation \( P \rtimes Q \) could refer to more than one group since there could be different actions \( Q \to \text{Aut}(P) \) leading to nonisomorphic semidirect products.
We will determine all these semidirect products, up to isomorphism, by working out all the ways \( \mathbb{Z}/(4) \) and \((\mathbb{Z}/(2))^2\) can act by automorphisms on \( \mathbb{Z}/(3) \) and all the ways \( \mathbb{Z}/(3) \) can act by automorphisms on \( \mathbb{Z}/(4) \) and \((\mathbb{Z}/(2))^2\).

First we list the automorphisms of the Sylow subgroups: \( \text{Aut}(\mathbb{Z}/(4)) \cong (\mathbb{Z}/(4))^\times = \{ \pm 1 \mod 4 \} \), \( \text{Aut}((\mathbb{Z}/(2))^2) \cong \text{GL}_2(\mathbb{Z}/(2)) \), and \( \text{Aut}(\mathbb{Z}/(3)) \cong (\mathbb{Z}/(3))^\times = \{ \pm 1 \mod 3 \} \).

**Case 1:** \( n_2 = 1, \ P \cong \mathbb{Z}/(4) \).

The 2-Sylow subgroup is normal, so the 3-Sylow subgroup acts on it. Our group is a semidirect product \( \mathbb{Z}/(4) \rtimes \mathbb{Z}/(3) \), for which the action of the second group on the first is through a homomorphism \( \varphi: \mathbb{Z}/(3) \to (\mathbb{Z}/(4))^\times \). The domain has order 3 and the target has order 2, so this homomorphism is trivial, and thus the semidirect product must be trivial: it’s the direct product

\[
\mathbb{Z}/(4) \times \mathbb{Z}/(3),
\]

which is cyclic of order 12 (generator \((1,1))\).

**Case 2:** \( n_2 = 1, \ P \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2) \).

We need to understand all homomorphisms \( \varphi: \mathbb{Z}/(3) \to \text{GL}_2(\mathbb{Z}/(2)) \). The trivial homomorphism leads to the direct product \((\mathbb{Z}/(2))^2 \times \mathbb{Z}/(3)\).

What about nontrivial homomorphisms \( \varphi: \mathbb{Z}/(3) \to \text{GL}_2(\mathbb{Z}/(2)) \)? Inside \( \text{GL}_2(\mathbb{Z}/(2)) \), which has order 6 (it’s isomorphic to \( S_3 \)), there is one subgroup of order 3: \( \{(1 0), (0 1), (1 1)\} \). A nontrivial homomorphism \( \varphi: \mathbb{Z}/(3) \to \text{GL}_2(\mathbb{Z}/(2)) \) is determined by where it sends 1 mod 3, which must go to a solution of \( A^3 = I_2 \); then \( \varphi(k \mod 3) = A^k \) in general. For \( \varphi \) to be nontrivial, \( A \) needs to have order 3, and there are two choices for that. The two matrices of order 3 in \( \text{GL}_2(\mathbb{Z}/(2)) \) are inverses. Call one of them \( A \), making the other \( A^{-1} \). The resulting homomorphisms \( \mathbb{Z}/(3) \to \text{GL}_2(\mathbb{Z}/(2)) \) are \( \varphi(k \mod 3) = A^k \) and \( \psi(k \mod 3) = A^{-k} \), which are related to each other by composition with inversion, but watch out: inversion is not an automorphism of \( \text{GL}_2(\mathbb{Z}/(2)) \). It is an automorphism of \( \mathbb{Z}/(3) \), where it’s negation. So precomposing \( \varphi \) with negation on \( \mathbb{Z}/(3) \) turns \( \varphi \) into \( \psi: \psi = \varphi \circ f \), where \( f(x) = -x \) on \( \mathbb{Z}/(3) \). Therefore the two nontrivial homomorphisms \( \mathbb{Z}/(3) \to \text{GL}_2(\mathbb{Z}/(2)) \) are linked through precomposition with an automorphism of \( \mathbb{Z}/(3) \), and therefore \( \varphi \) and \( \psi \) define isomorphic semidirect products. This means that up to isomorphism, there is one nontrivial semidirect product

\[
(\mathbb{Z}/(2))^2 \times \mathbb{Z}/(3).
\]

That is, we have shown that up to isomorphism there is only one group of order 12 with \( n_2 = 1 \) and 2-Sylow subgroup isomorphic to \( \mathbb{Z}/(2) \times \mathbb{Z}/(2) \). The group \( A_4 \) fits this description: its normal 2-Sylow subgroup is \( \{(1), (12)(34), (13)(24), (14)(23)\} \), which is not cyclic.

Now assume \( n_2 \neq 1 \), so \( n_2 = 3 \) and \( n_3 = 1 \). Since \( n_2 > 1 \), the group is nonabelian, so it’s a nontrivial semidirect product (a direct product of abelian groups is abelian).

**Case 3:** \( n_2 = 3, \ n_3 = 1 \), and \( P \cong \mathbb{Z}/(4) \).

Our group looks like \( \mathbb{Z}/(3) \rtimes \mathbb{Z}/(4) \), built from a nontrivial homomorphism \( \varphi: \mathbb{Z}/(4) \to \text{Aut}(\mathbb{Z}/(3)) = (\mathbb{Z}/(3))^\times \) There is only one choice of \( \varphi \): it has to send 1 mod 4 to \(-1 \mod 3 \), which determines everything else: \( \varphi(c \mod 4) = (-1)^c \mod 3 \). Therefore there is one
nontrivial semidirect product \( \mathbb{Z}/(3) \rtimes \mathbb{Z}/(4) \). Explicitly, this group is the set \( \mathbb{Z}/(3) \times \mathbb{Z}/(4) \) with group law

\[
(a, b)(c, d) = (a + (-1)^b c, b + d).
\]

Case 4: \( n_2 = 3, n_3 = 1, \) and \( P \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2) \).

The group is \( \mathbb{Z}/(3) \rtimes (\mathbb{Z}/(2))^2 \) for a nontrivial homomorphism \( \varphi : (\mathbb{Z}/(2))^2 \to (\mathbb{Z}/(3))^\times \).

The group \( (\mathbb{Z}/(2))^2 \) has a pair of generators \((1, 0)\) and \((0, 1)\), and \( \varphi(a, b) = \varphi(1, 0)^a \varphi(0, 1)^b \), where \( \varphi(1, 0) \) and \( \varphi(0, 1) \) are \( \pm 1 \). Conversely, this formula for \( \varphi \) defines a homomorphism since \( a \) and \( b \) are in \( \mathbb{Z}/(2) \) and exponents on \( \pm 1 \) only matter mod 2. For \( \varphi \) to be nontrivial means \( \varphi(1, 0) \) and \( \varphi(0, 1) \) are not both 1, so there are three choices of \( \varphi : (\mathbb{Z}/(2))^2 \to (\mathbb{Z}/(3))^\times \):

\[
\varphi(a, b) = (-1)^a, \quad \varphi(a, b) = (-1)^b, \quad \varphi(a, b) = (-1)^a(-1)^b = (-1)^{a+b}.
\]

This does not mean the three corresponding semidirect products \( \mathbb{Z}/(3) \rtimes \varphi (\mathbb{Z}/(2))^2 \) are nonisomorphic. In fact, the above three choices of \( \varphi \) lead to isomorphic semidirect products: precomposing the first \( \varphi \) with the matrix \( (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \) produces the second \( \varphi \), and precomposing the first \( \varphi \) with the matrix \( (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \) produces the third \( \varphi \). Therefore the three nontrivial semidirect products \( \mathbb{Z}/(3) \rtimes (\mathbb{Z}/(2))^2 \) are isomorphic, so all groups of order 12 with \( n_2 = 3 \) (equivalently, all nonabelian groups of order 12 with \( n_3 = 1 \)) and 2-Sylow subgroup isomorphic to \( (\mathbb{Z}/(2))^2 \) are isomorphic. One such group is \( D_6 \), with normal 3-Sylow subgroup \( \{1, r^2, r^4\} \).

If we meet a group \( G \) of order 12, then we can decide which of the 5 groups \( G \) is isomorphic to by the following procedure:

- If \( G \) is abelian then \( G \) is isomorphic to \( \mathbb{Z}/(4) \times \mathbb{Z}/(3) \cong \mathbb{Z}/(12) \) or \( \mathbb{Z}/(2) \times \mathbb{Z}/(2) \times \mathbb{Z}/(3) \), which are distinguished by the structure of the 2-Sylow subgroup.
- If \( G \) is nonabelian then \( G \cong A_4 \) if \( n_2 = 1 \) (a normal 2-Sylow subgroup) or \( n_3 > 1 \).
- If \( G \) is nonabelian then \( G \cong D_6 \) if \( n_2 > 1 \) and its 2-Sylow subgroups are noncyclic or if \( G \) has more than three elements of order 2.
- If \( G \) is nonabelian then \( G \) is isomorphic to the nontrivial semidirect product \( \mathbb{Z}/(3) \rtimes \mathbb{Z}/(4) \) if \( n_2 > 1 \) and its 2-Sylow subgroup is cyclic or if \( G \) has an element of order 4.

For example, here are five groups of order 12:

\[ \mathbb{Z}/(2) \times \mathbb{Z}/(6), \quad \mathbb{Z}/(2) \times S_3, \quad \text{PSL}_2(F_3), \quad \text{Aff}(\mathbb{Z}/(6)), \quad \text{Aff}(F_4). \]

The first group is abelian, and \( \mathbb{Z}/(2) \times \mathbb{Z}/(6) \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2) \times \mathbb{Z}/(3) \) by the Chinese remainder theorem. The remaining groups are nonabelian. Since \( \mathbb{Z}/(2) \times S_3 \) has more than three elements of order 2, \( \mathbb{Z}/(2) \times S_3 \cong D_6 \). The group \( \text{PSL}_2(F_3) \) is nonabelian with \( n_2 = 1 \), so \( \text{PSL}_2(F_3) \cong A_4 \). The group \( \text{Aff}(\mathbb{Z}/(6)) \) has \( n_2 > 1 \) and its 2-Sylow subgroups are not cyclic, so \( \text{Aff}(\mathbb{Z}/(6)) \cong D_6 \). Finally, \( \text{Aff}(F_4) \) is nonabelian with more than one subgroup of order 3, so \( \text{Aff}(F_4) \cong A_4 \).

Another way to distinguish between groups of order 12 is by counting elements of a certain order. From the table below, these groups can be distinguished by counting elements of order 2 except for \((\mathbb{Z}/(2))^2 \times \mathbb{Z}/(4)\) and \( A_4 \), where one is abelian and the other isn’t.

<table>
<thead>
<tr>
<th>Group</th>
<th>Order 1</th>
<th>Order 2</th>
<th>Order 3</th>
<th>Order 4</th>
<th>Order 6</th>
<th>Order 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}/(12) )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>((\mathbb{Z}/(2))^2 \times \mathbb{Z}/(3))</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>1</td>
<td>3</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( D_6 )</td>
<td>1</td>
<td>7</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>( \mathbb{Z}/(3) \times \mathbb{Z}/(4) )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>
In abstract algebra textbooks (not group theory textbooks), $\mathbb{Z}/(3) \rtimes \mathbb{Z}/(4)$ is usually written as $T$ but it is almost never given a name to accompany the label. Should it be called the “obscure group of order 12”? Actually, this group belongs to a standard family of finite groups: the dicyclic groups, also called the binary dihedral groups. They are nonabelian with order $4n$ ($n \geq 2$) and each contains a unique element of order 2. The one of order 8 is $Q_8$, and more generally the one of order $2^m$ is the generalized quaternion group $Q_{2^m}$.

We said at the start that the list of groups of order 12 first given by Kempe [2, pp. 37–43] had a mistake. For his 5 proposed groups, written in tabular form (a list of permutations in $S_{12}$) and called $T_1, T_2, \ldots, T_5$, it turns out that $T_1 \cong \mathbb{Z}/(12)$, $T_2 \cong (\mathbb{Z}/(2))^2 \times \mathbb{Z}/(3)$, $T_3 \cong D_6$, and $T_5 \cong A_4$, but his $T_4$ (given below) is not a group and Kempe’s list did not include $\mathbb{Z}/(3) \rtimes \mathbb{Z}/(4)$. The permutations in $T_4$ are a subset of $S_{12}$ but not a subgroup of $S_{12}$. The rows are each permutations of $a, b, \ldots, k, l$ and there are two 12-cycles (the 7th and 11th rows) but there is no permutation of order 6 and most rows that are not permutations of order 2 do not have their inverse in the table (e.g., rows 3 and 4).

\begin{verbatim}
 a b c d e f g h i j k l
 b a d c j i l k f e h g
 c d b a g h f e k l j i
 d c a b l k i j h g e f
 e f g h i j k l a b c d
 f e h g b a d c j i l k
 g h f e k l j i c d b a
 h g e f d c a b l k i j
 i j k l a b c d e f g h
 j i l k f e h g b a d c
 k l j i c d b a g h f e
 l k i j h g e f d c a b
\end{verbatim}

References
