GROUPS OF ORDER 12

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The groups of order 12, up to isomorphism, were first determined in the 1880s by Cayley [1] and Kempe [2, pp. 37–43]: Kempe gave a list of 5 groups and Cayley pointed out a few years later that one of Kempe's groups did not make sense and that Kempe had missed an example, which Cayley provided.

We will use semidirect products to describe all groups of order 12 up to isomorphism. There turn out to be 5 such groups: 2 are abelian and 3 are nonabelian. The nonabelian groups are an alternating group, a dihedral group, and a third less familiar group.

Theorem 1. *Every group of order* 12 *is a semidirect product of a group of order* 3 *and a group of order* 4.

Proof. Let $|G| = 12 = 2^2 \cdot 3$. A 2-Sylow subgroup has order 4 and a 3-Sylow subgroup has order 3. We will start by showing G has a normal 2-Sylow subgroup or a normal 3-Sylow subgroup: $n_2 = 1$ or $n_3 = 1$. From the Sylow theorems,

$$n_2 \mid 3, \quad n_2 \equiv 1 \bmod 2, \quad n_3 \mid 4, \quad n_3 \equiv 1 \bmod 3.$$

Therefore $n_2 = 1$ or 3 and $n_3 = 1$ or 4.

To show $n_2 = 1$ or $n_3 = 1$, assume $n_3 \neq 1$. Then $n_3 = 4$. Let's count elements of order 3. Since each 3-Sylow subgroup has prime order 3, two different 3-Sylow subgroups intersect trivially. Each of the four 3-Sylow subgroups of G contains two elements of order 3 and they are not in another 3-Sylow subgroup, so the number of elements in G of order 3 is $2n_2 = 8$. This leaves us with $12 - 8 = 4$ elements in G not of order 3. A 2-Sylow subgroup has order 4 and contains no elements of order 3, so one 2-Sylow subgroup must account for the remaining 4 elements of G. Thus $n_2 = 1$ if $n_3 \neq 1$.

Next we show G is a semidirect product of a 2-Sylow and 3-Sylow subgroup. Let P_2 be a 2-Sylow subgroup and P_3 be a 3-Sylow subgroup of G. Since P_2 and P_3 have relatively prime orders, $P_2 \cap P_3 = \{1\}$ and the set $P_2P_3 = \{xy : x \in P_2, y \in P_3\}$ has size $|P_2||P_3|/|P_2 \cap P_3| = 12 = |G|$, so $G = P_2P_3$. Since P_2 or P_3 is normal in G, G is a semidirect product of P_2 and P_3: $G \cong P_2 \rtimes P_3$ if $P_2 \lhd G$ and $G \cong P_3 \rtimes P_2$ if $P_3 \lhd G$.[1] $\square$

Groups of order 4 are isomorphic to $\mathbf{Z}/(4)$ or $(\mathbf{Z}/(2))^2$, and groups of order 3 are isomorphic to $\mathbf{Z}/(3)$, so every group of order 12 is a semidirect product of the form

$$\mathbf{Z}/(4) \rtimes \mathbf{Z}/(3), \quad (\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(3), \quad \mathbf{Z}/(3) \rtimes \mathbf{Z}/(4), \quad \mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2.$$

We will determine all these semidirect products, up to isomorphism, by working out all the ways $\mathbf{Z}/(4)$ and $(\mathbf{Z}/(2))^2$ can act by automorphisms on $\mathbf{Z}/(3)$ and all the ways $\mathbf{Z}/(3)$ can act by automorphisms on $\mathbf{Z}/(4)$ and $(\mathbf{Z}/(2))^2$.

Theorem 2. *Every group of order* 12 *is isomorphic to one of* $\mathbf{Z}/(12)$, $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$, A_4, D_6, *or the nontrivial semidirect product* $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$.

[1] The notation $P_2 \rtimes P_3$ could refer to more than one group since there could be different actions $P_3 \to \mathrm{Aut}(P_2)$ leading to nonisomorphic semidirect products.

We say *the* nontrivial semidirect product $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ since there is only one nontrivial homomorphism $\mathbf{Z}/(4) \to \mathrm{Aut}(\mathbf{Z}/(3)) = (\mathbf{Z}/(3))^\times$, namely $k \bmod 4 \mapsto (-1)^k \bmod 3$. The corresponding group $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ has group law

$$(a,b)(c,d) = (a + (-1)^b c, b + d). \tag{1}$$

As an abstract group, this is generated by $x = (1,0)$ and $y = (0,1)$ with $x^3 = 1$, $y^4 = 1$, and $yxy^{-1} = x^{-1}$. A concrete model for this group inside $\mathrm{SL}_2(\mathbf{C})$ has $x = \left(\begin{smallmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{smallmatrix}\right)$ with $\omega = e^{2\pi i/3}$ and $y = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$.

In the proof of Theorem 2, we will appeal to an isomorphism property of semidirect products: for each semidirect product $H \rtimes_\varphi K$ and automorphism $f \colon K \to K$, $H \rtimes_\varphi K \cong H \rtimes_{\varphi \circ f} K$. This says that precomposing an action of K on H by automorphisms (that's φ) with an automorphism of K produces an isomorphic semidirect product of H and K.

Proof. First we list the automorphisms of the possible Sylow subgroups: $\mathrm{Aut}(\mathbf{Z}/(4)) \cong (\mathbf{Z}/(4))^\times = \{\pm 1 \bmod 4\}$, $\mathrm{Aut}((\mathbf{Z}/(2))^2) \cong \mathrm{GL}_2(\mathbf{Z}/(2))$, and $\mathrm{Aut}(\mathbf{Z}/(3)) \cong (\mathbf{Z}/(3))^\times = \{\pm 1 \bmod 3\}$.

Case 1: $n_2 = 1$, $P_2 \cong \mathbf{Z}/(4)$.

The 2-Sylow subgroup is normal, so the 3-Sylow subgroup acts on it. Our group is a semidirect product $\mathbf{Z}/(4) \rtimes \mathbf{Z}/(3)$, for which the action of the second group on the first is through a homomorphism $\varphi \colon \mathbf{Z}/(3) \to (\mathbf{Z}/(4))^\times$. The domain has order 3 and the target has order 2, so this homomorphism is trivial, and thus the semidirect product must be trivial: it's the direct product

$$\mathbf{Z}/(4) \times \mathbf{Z}/(3),$$

which is cyclic of order 12 (generator $(1,1)$).

Case 2: $n_2 = 1$, $P_2 \cong \mathbf{Z}/(2) \times \mathbf{Z}/(2)$.

We need to understand all homomorphisms $\varphi \colon \mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$. The trivial homomorphism leads to the direct product

$$(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3).$$

What about nontrivial homomorphisms $\varphi \colon \mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$? Inside $\mathrm{GL}_2(\mathbf{Z}/(2))$, which has order 6 (it's isomorphic to S_3), there is one subgroup of order 3: $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}$. A nontrivial homomorphism $\varphi \colon \mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$ is determined by where it sends $1 \bmod 3$, which must go to a solution of $A^3 = I_2$; then $\varphi(k \bmod 3) = A^k$ in general. For φ to be nontrivial, A needs to have order 3, and there are two choices for that. The two matrices of order 3 in $\mathrm{GL}_2(\mathbf{Z}/(2))$ are inverses. Call one of them A, making the other A^{-1}. The resulting homomorphisms $\mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$ are $\varphi(k \bmod 3) = A^k$ and $\psi(k \bmod 3) = A^{-k}$, which are related to each other by composition with inversion, but *watch out*: inversion is not an automorphism of $\mathrm{GL}_2(\mathbf{Z}/(2))$. It is an automorphism of $\mathbf{Z}/(3)$, where it's negation. So precomposing φ with negation on $\mathbf{Z}/(3)$ turns φ into ψ: $\psi = \varphi \circ f$, where $f(x) = -x$ on $\mathbf{Z}/(3)$. Therefore the two nontrivial homomorphisms $\mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$ are linked through precomposition with an automorphism of $\mathbf{Z}/(3)$, and therefore φ and ψ define isomorphic semidirect products. This means that up to isomorphism, there is one nontrivial semidirect product

$$(\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(3).$$

That is, we have shown that up to isomorphism there is only one group of order 12 with $n_2 = 1$ and 2-Sylow subgroup isomorphic to $\mathbf{Z}/(2) \times \mathbf{Z}/(2)$. The group A_4 fits this description: its normal 2-Sylow subgroup is $\{(1), (12)(34), (13)(24), (14)(23)\}$, which is not cyclic.

Now assume $n_2 \neq 1$, so $n_2 = 3$ and $n_3 = 1$. Since $n_2 > 1$, the group is nonabelian, so it's a nontrivial semidirect product (a direct product of abelian groups is abelian).

Case 3: $n_2 = 3$, $n_3 = 1$, and $P_2 \cong \mathbf{Z}/(4)$.

Our group looks like $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$, built from a nontrivial homomorphism $\varphi\colon \mathbf{Z}/(4) \to \mathrm{Aut}(\mathbf{Z}/(3)) = (\mathbf{Z}/(3))^\times$ There is only one choice of φ: it has to send $1 \bmod 4$ to $-1 \bmod 3$, which determines everything else: $\varphi(c \bmod 4) = (-1)^c \bmod 3$. Therefore there is one nontrivial semidirect product $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ and its group operation is given by (1).

Case 4: $n_2 = 3$, $n_3 = 1$, and $P_2 \cong \mathbf{Z}/(2) \times \mathbf{Z}/(2)$.

The group is $\mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2$ for a nontrivial homomorphism $\varphi\colon (\mathbf{Z}/(2))^2 \to (\mathbf{Z}/(3))^\times$. The group $(\mathbf{Z}/(2))^2$ has a pair of generators $(1,0)$ and $(0,1)$, and $\varphi(a,b) = \varphi(1,0)^a\varphi(0,1)^b$, where $\varphi(1,0)$ and $\varphi(0,1)$ are ± 1. Conversely, this formula for φ defines a homomorphism since a and b are in $\mathbf{Z}/(2)$ and exponents on ± 1 only matter mod 2. For φ to be nontrivial means $\varphi(1,0)$ and $\varphi(0,1)$ are not both 1, so there are three choices of $\varphi\colon (\mathbf{Z}/(2))^2 \to (\mathbf{Z}/(3))^\times$:

$$\varphi(a,b) = (-1)^a, \quad \varphi(a,b) = (-1)^b, \quad \varphi(a,b) = (-1)^a(-1)^b = (-1)^{a+b}.$$

This does *not* mean the three corresponding semidirect products $\mathbf{Z}/(3) \rtimes_\varphi (\mathbf{Z}/(2))^2$ are nonisomorphic. In fact, the above three choices of φ lead to isomorphic semidirect products: precomposing the first φ with the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ produces the second φ, and precomposing the first φ with the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ produces the third φ. Therefore the three nontrivial semidirect products $\mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2$ are isomorphic, so all groups of order 12 with $n_2 = 3$ (equivalently, all nonabelian groups of order 12 with $n_3 = 1$) and 2-Sylow subgroup isomorphic to $(\mathbf{Z}/(2))^2$ are isomorphic. One such group is D_6, with normal 3-Sylow subgroup $\{1, r^2, r^4\}$. □

If we meet a group G of order 12, then we can decide which of the 5 groups G is isomorphic to by the following procedure:

- If G is abelian then G is isomorphic to $\mathbf{Z}/(4) \times \mathbf{Z}(3) \cong \mathbf{Z}/(12)$ or G is isomorphic to $\mathbf{Z}/(2) \times \mathbf{Z}/(2) \times \mathbf{Z}/(3)$. These options are distinguished by their 2-Sylow subgroup.
- If G is nonabelian then $G \cong A_4$ if $n_2 = 1$ (a normal 2-Sylow subgroup) or $n_3 > 1$.
- If G is nonabelian then $G \cong D_6$ if $n_2 > 1$ and its 2-Sylow subgroups are noncyclic or if G has more than three elements of order 2.
- If G is nonabelian then G is isomorphic to the nontrivial semidirect product $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ if $n_2 > 1$ and its 2-Sylow subgroup is cyclic or if G has an element of order 4.

For example, here are five groups of order 12:

$$\mathbf{Z}/(2) \times \mathbf{Z}(6), \quad \mathbf{Z}/(2) \times S_3, \quad \mathrm{PSL}_2(\mathbf{F}_3), \quad \mathrm{Aff}(\mathbf{Z}/(6)), \quad \mathrm{Aff}(\mathbf{F}_4).$$

The first group is abelian, and $\mathbf{Z}/(2) \times \mathbf{Z}(6) \cong \mathbf{Z}/(2) \times \mathbf{Z}/(2) \times \mathbf{Z}/(3)$ by the Chinese remainder theorem. The remaining groups are nonabelian. Since $\mathbf{Z}/(2) \times S_3$ has more than three elements of order 2, $\mathbf{Z}/(2) \times S_3 \cong D_6$. The group $\mathrm{PSL}_2(\mathbf{F}_3)$ is nonabelian with $n_2 = 1$, so $\mathrm{PSL}_2(\mathbf{F}_3) \cong A_4$. The group $\mathrm{Aff}(\mathbf{Z}/(6))$ has $n_2 > 1$ and its 2-Sylow subgroups are not cyclic, so $\mathrm{Aff}(\mathbf{Z}/(6)) \cong D_6$. Finally, $\mathrm{Aff}(\mathbf{F}_4)$ is nonabelian with more than one subgroup of order 3, so $\mathrm{Aff}(\mathbf{F}_4) \cong A_4$.

Another way to distinguish between groups of order 12 is by counting elements of a certain order. From Table 1 below, these groups can be distinguished by counting elements of order 2 except for $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(4)$ and A_4, where one is abelian and the other isn't.

In abstract algebra textbooks (not group theory textbooks), $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ is often written as T but it is almost never given a name to accompany the label. Should it be called the

Group	Order 1	Order 2	Order 3	Order 4	Order 6	Order 12
$\mathbf{Z}/(12)$	1	1	2	2	2	4
$(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$	1	3	2	0	6	0
A_4	1	3	8	0	0	0
D_6	1	7	2	0	2	0
$\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$	1	1	2	6	2	0

TABLE 1. Counting orders of elements in groups of order 12.

"obscure group of order 12"? Actually, this group belongs to a standard family of finite groups: the dicyclic groups, also called the binary dihedral groups. They are nonabelian with order $4n$ ($n \geq 2$) and each contains a unique element of order 2. In $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$, the unique element of order 2 is $(0, 2)$. The dicyclic group of order 8 is Q_8, and more generally the dicyclic of order 2^m is the generalized quaternion group Q_{2^m}.

We said at the start that the list of groups of order 12 first given by Kempe [2, pp. 37–43] has a mistake. Kempe wrote each of his 5 proposed groups in tabular form (as a list of 12 permutations in S_{12}) and called them $T_1, T_2, \ldots, T_5$. It turns out that $T_1 \cong \mathbf{Z}/(12)$, $T_2 \cong (\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$, $T_3 \cong D_6$, and $T_5 \cong A_4$, but Kempe's T_4, which is given below in Table 2, is not a group and his list of groups of order 12 did not include a group isomorphic to $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$. The permutations in T_4 are a subset of S_{12} but not a subgroup of S_{12}. Each row is a permutation of $a, b, \ldots, k, l$ and two of the rows are 12-cycles (the 7th and 11th rows) but no row is a permutation of order 6 and most rows that have order greater than 2 do not have their inverse in the table (*e.g.*, rows 3 and 4).

a	b	c	d	e	f	g	h	i	j	k	l
b	a	d	c	j	i	l	k	f	e	h	g
c	d	b	a	g	h	f	e	k	l	j	i
d	c	a	b	l	k	i	j	h	g	e	f
e	f	g	h	i	j	k	l	a	b	c	d
f	e	h	g	b	a	d	c	j	i	l	k
g	h	f	e	k	l	j	i	c	d	b	a
h	g	e	f	d	c	a	b	l	k	i	j
i	j	k	l	a	b	c	d	e	f	g	h
j	i	l	k	f	e	h	g	b	a	d	c
k	l	j	i	c	d	b	a	g	h	f	e
l	k	i	j	h	g	e	f	d	c	a	b

TABLE 2. Kempe's false group of order 12.

REFERENCES

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