

GROUPS OF ORDER 12

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The groups of order 12, up to isomorphism, were first determined in the 1880s: Kempe [3, pp. 37–43] gave a list of 5 groups and Cayley [1] pointed out a few years later that one of Kempe’s groups did not make sense and that a specific group was missed.

We will use semidirect products to describe all 5 groups of order 12 up to isomorphism. Two are abelian and the others are A_4 , D_6 , and a less familiar group.

Theorem 1. *Every group of order 12 is a semidirect product of a group of order 3 and a group of order 4.*

Proof. Let $|G| = 12 = 2^2 \cdot 3$. A 2-Sylow subgroup has order 4 and a 3-Sylow subgroup has order 3. We will start by showing G has a normal 2-Sylow subgroup or a normal 3-Sylow subgroup: $n_2 = 1$ or $n_3 = 1$. From the Sylow theorems,

$$n_2 \mid 3, \quad n_2 \equiv 1 \pmod{2}, \quad n_3 \mid 4, \quad n_3 \equiv 1 \pmod{3}.$$

Therefore $n_2 = 1$ or 3 and $n_3 = 1$ or 4.

To show $n_2 = 1$ or $n_3 = 1$, assume $n_3 \neq 1$. Then $n_3 = 4$. Let’s count elements of order 3. Since each 3-Sylow subgroup has order 3, different 3-Sylow subgroups intersect trivially. Each of the 3-Sylow subgroups of G contains two elements of order 3, so the number of elements in G of order 3 is $2n_3 = 8$. This leaves us with $12 - 8 = 4$ elements in G not of order 3. A 2-Sylow subgroup has order 4 and contains no elements of order 3, so one 2-Sylow subgroup must account for the remaining 4 elements of G . Thus $n_2 = 1$ if $n_3 \neq 1$.

Next we show G is a semidirect product of a 2-Sylow and 3-Sylow subgroup. Let P_2 be a 2-Sylow subgroup and P_3 be a 3-Sylow subgroup of G . Since P_2 and P_3 have relatively prime orders, $P_2 \cap P_3 = \{1\}$ and the set $P_2P_3 = \{xy : x \in P_2, y \in P_3\}$ has size $|P_2||P_3|/|P_2 \cap P_3| = 12 = |G|$, so $G = P_2P_3$. Since P_2 or P_3 is normal in G , G is a semidirect product of P_2 and P_3 : $G \cong P_2 \rtimes P_3$ if $P_2 \triangleleft G$ and $G \cong P_3 \rtimes P_2$ if $P_3 \triangleleft G$.¹ \square

Groups of order 4 are isomorphic to $\mathbf{Z}/(4)$ or $(\mathbf{Z}/(2))^2$, and groups of order 3 are isomorphic to $\mathbf{Z}/(3)$, so every group of order 12 is a semidirect product of the form

$$\mathbf{Z}/(4) \rtimes \mathbf{Z}/(3), \quad (\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(3), \quad \mathbf{Z}/(3) \rtimes \mathbf{Z}/(4), \quad \mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2.$$

To determine these up to isomorphism, we work out how $\mathbf{Z}/(4)$ and $(\mathbf{Z}/(2))^2$ act by automorphisms on $\mathbf{Z}/(3)$ and how $\mathbf{Z}/(3)$ acts by automorphisms on $\mathbf{Z}/(4)$ and $(\mathbf{Z}/(2))^2$.

Theorem 2. *Every group of order 12 is isomorphic to one of $\mathbf{Z}/(12)$, $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$, A_4 , D_6 , or the nontrivial semidirect product $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$.*

We say *the* nontrivial semidirect product $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ since there is only one nontrivial homomorphism $\mathbf{Z}/(4) \rightarrow \text{Aut}(\mathbf{Z}/(3)) = (\mathbf{Z}/(3))^\times$, namely $k \pmod{4} \mapsto (-1)^k \pmod{3}$. The

¹The notation $P_2 \rtimes P_3$ could refer to more than one group since there could be different actions $P_3 \rightarrow \text{Aut}(P_2)$ leading to nonisomorphic semidirect products.

corresponding group $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ has group law

$$(1) \quad (a, b)(c, d) = (a + (-1)^b c, b + d).$$

This is generated by $x = (1, 0)$ and $y = (0, 1)$ with $x^3 = 1$, $y^4 = 1$, and $xyx^{-1} = x^{-1}$. A model for this group inside $\mathrm{SL}_2(\mathbf{C})$ has $x = \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}$ with $\omega = e^{2\pi i/3}$ and $y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

In the proof of Theorem 2, we will appeal to an isomorphism property of semidirect products: for each semidirect product $H \rtimes_{\varphi} K$ and automorphism $f: K \rightarrow K$, $H \rtimes_{\varphi} K \cong H \rtimes_{\varphi \circ f} K$. This says that precomposing an action of K on H by automorphisms (that's φ) with an automorphism of K produces an isomorphic semidirect product of H and K .

Proof. Here are automorphisms of possible Sylow subgroups: $\mathrm{Aut}(\mathbf{Z}/(4)) \cong (\mathbf{Z}/(4))^{\times} = \{\pm 1 \bmod 4\}$, $\mathrm{Aut}((\mathbf{Z}/(2))^2) \cong \mathrm{GL}_2(\mathbf{Z}/(2))$, and $\mathrm{Aut}(\mathbf{Z}/(3)) \cong (\mathbf{Z}/(3))^{\times} = \{\pm 1 \bmod 3\}$.

Case 1: $n_2 = 1$, $P_2 \cong \mathbf{Z}/(4)$.

The 2-Sylow subgroup is normal, so the 3-Sylow subgroup acts on it. Our group is a semidirect product $\mathbf{Z}/(4) \rtimes \mathbf{Z}/(3)$, for which the action of the second group on the first is through a homomorphism $\varphi: \mathbf{Z}/(3) \rightarrow (\mathbf{Z}/(4))^{\times}$. The domain has order 3 and the target has order 2, so this homomorphism is trivial, and thus the semidirect product must be trivial: it's the direct product

$$\mathbf{Z}/(4) \times \mathbf{Z}/(3),$$

which is cyclic of order 12 (generator $(1, 1)$).

Case 2: $n_2 = 1$, $P_2 \cong \mathbf{Z}/(2) \times \mathbf{Z}/(2)$.

We need to understand all homomorphisms $\varphi: \mathbf{Z}/(3) \rightarrow \mathrm{GL}_2(\mathbf{Z}/(2))$. The trivial homomorphism leads to the direct product

$$(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3).$$

What about nontrivial homomorphisms $\varphi: \mathbf{Z}/(3) \rightarrow \mathrm{GL}_2(\mathbf{Z}/(2))$? Inside $\mathrm{GL}_2(\mathbf{Z}/(2))$, which has order 6 (it's isomorphic to S_3), there is one subgroup of order 3: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$. A nontrivial homomorphism $\varphi: \mathbf{Z}/(3) \rightarrow \mathrm{GL}_2(\mathbf{Z}/(2))$ is determined by where it sends $1 \bmod 3$, which must go to a solution of $A^3 = I_2$; then $\varphi(k \bmod 3) = A^k$ in general. For φ to be nontrivial, A needs to have order 3, and there are two choices for that. The two matrices of order 3 in $\mathrm{GL}_2(\mathbf{Z}/(2))$ are inverses. Call one of them A , making the other A^{-1} . The resulting homomorphisms $\mathbf{Z}/(3) \rightarrow \mathrm{GL}_2(\mathbf{Z}/(2))$ are $\varphi(k \bmod 3) = A^k$ and $\psi(k \bmod 3) = A^{-k}$, which are related to each other by composition with inversion, but *watch out*: inversion is not an automorphism of $\mathrm{GL}_2(\mathbf{Z}/(2))$. It is an automorphism of $\mathbf{Z}/(3)$, where it's negation. So precomposing φ with negation on $\mathbf{Z}/(3)$ turns φ into ψ : $\psi = \varphi \circ f$, where $f(x) = -x$ on $\mathbf{Z}/(3)$. Therefore the two nontrivial homomorphisms $\mathbf{Z}/(3) \rightarrow \mathrm{GL}_2(\mathbf{Z}/(2))$ are linked through precomposition with an automorphism of $\mathbf{Z}/(3)$, so φ and ψ define isomorphic semidirect products. Thus up to isomorphism, there is one nontrivial semidirect product

$$(\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(3).$$

Since up to isomorphism one group of order 12 has $n_2 = 1$ and a noncyclic 2-Sylow subgroup, and A_4 also fits this description, this semidirect product is isomorphic to A_4 .

Now assume $n_2 \neq 1$, so $n_2 = 3$ and $n_3 = 1$. Since $n_2 > 1$, the group is nonabelian, so it's a nontrivial semidirect product (a direct product of abelian groups is abelian).

Case 3: $n_2 = 3$, $n_3 = 1$, and $P_2 \cong \mathbf{Z}/(4)$.

Our group looks like $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$, built from a nontrivial homomorphism $\varphi: \mathbf{Z}/(4) \rightarrow \mathrm{Aut}(\mathbf{Z}/(3)) = (\mathbf{Z}/(3))^{\times}$. There is only one choice of φ : it has to send $1 \bmod 4$ to $-1 \bmod 3$

3, which determines everything else: $\varphi(c \bmod 4) = (-1)^c \bmod 3$. Therefore there is one nontrivial semidirect product $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ and its group operation is given by (1).

Case 4: $n_2 = 3$, $n_3 = 1$, and $P_2 \cong \mathbf{Z}/(2) \times \mathbf{Z}/(2)$.

The group is $\mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2$ for a nontrivial homomorphism $\varphi: (\mathbf{Z}/(2))^2 \rightarrow (\mathbf{Z}/(3))^\times$. The group $(\mathbf{Z}/(2))^2$ has a pair of generators $(1, 0)$ and $(0, 1)$, and $\varphi(a, b) = \varphi(1, 0)^a \varphi(0, 1)^b$, where $\varphi(1, 0)$ and $\varphi(0, 1)$ are ± 1 . Conversely, this formula for φ defines a homomorphism since a and b are in $\mathbf{Z}/(2)$ and exponents on ± 1 only matter mod 2. For φ to be nontrivial means $\varphi(1, 0)$ and $\varphi(0, 1)$ are not both 1, so there are three choices of $\varphi: (\mathbf{Z}/(2))^2 \rightarrow (\mathbf{Z}/(3))^\times$:

$$\varphi(a, b) = (-1)^a, \quad \varphi(a, b) = (-1)^b, \quad \varphi(a, b) = (-1)^a(-1)^b = (-1)^{a+b}.$$

This does *not* mean the three corresponding semidirect products $\mathbf{Z}/(3) \rtimes_{\varphi} (\mathbf{Z}/(2))^2$ are nonisomorphic. In fact, the above three choices of φ lead to isomorphic semidirect products: precomposing the first φ with the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ produces the second φ , and precomposing the first φ with the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ produces the third φ . Therefore the three nontrivial semidirect products $\mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2$ are isomorphic, so all groups of order 12 with $n_2 = 3$ (equivalently, all nonabelian groups of order 12 with $n_3 = 1$) and 2-Sylow subgroup isomorphic to $(\mathbf{Z}/(2))^2$ are isomorphic. One such group is D_6 , with normal 3-Sylow subgroup $\{1, r^2, r^4\}$. \square

For a group of order 12, Table 1 lists structural properties to know it up to isomorphism. (That $n_3 = 4$ implies $G \cong A_4$ is because G acting by conjugation on its 4 3-Sylow subgroups is an isomorphism of G with A_4 .)

Group	Abelian?	n_2	n_3	2-Sylow
$\mathbf{Z}/(12)$	Yes	1	1	cyclic
$(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$	Yes	1	1	noncyclic
A_4	No	1	4	noncyclic
D_6	No	3	1	noncyclic
$\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$	No	3	1	cyclic

TABLE 1. Structural properties of groups of order 12.

For example, here are five groups of order 12:

$$(2) \quad \mathbf{Z}/(2) \times \mathbf{Z}/(6), \quad \mathbf{Z}/(2) \times S_3, \quad \mathrm{PSL}_2(\mathbf{F}_3), \quad \mathrm{Aff}(\mathbf{Z}/(6)), \quad \mathrm{Aff}(\mathbf{F}_4).$$

The first group is abelian with noncyclic 2-Sylow subgroup, so it's isomorphic to $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$ (or use the Chinese remainder theorem). The remaining groups are nonabelian. Since $\mathbf{Z}/(2) \times S_3$ has $n_3 = 1$ and a noncyclic 2-Sylow subgroup, $\mathbf{Z}/(2) \times S_3 \cong D_6$. The group $\mathrm{PSL}_2(\mathbf{F}_3)$ has $n_3 > 1$, so $\mathrm{PSL}_2(\mathbf{F}_3) \cong A_4$. The group $\mathrm{Aff}(\mathbf{Z}/(6))$ has $n_2 > 1$ and noncyclic 2-Sylow subgroup, so $\mathrm{Aff}(\mathbf{Z}/(6)) \cong D_6$. Finally, $\mathrm{Aff}(\mathbf{F}_4)$ has $n_3 > 1$, so $\mathrm{Aff}(\mathbf{F}_4) \cong A_4$.

Another way to distinguish between groups of order 12 is by counting elements of a certain order. From Table 2 below, these groups can be distinguished by counting elements of order 2 except for $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(4)$ and A_4 , where one is abelian and the other isn't.

For example, among the groups in (2), $\mathbf{Z}/(2) \times \mathbf{Z}/(6)$ is abelian with three elements of order 2, so it is isomorphic to $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$. Since $\mathbf{Z}/(2) \times S_3$ has more than 3 elements of order 2, it is isomorphic to D_6 . Since $\mathrm{PSL}_2(\mathbf{F}_3)$ has more than 2 elements of order 3, it is isomorphic to A_4 . Since $\mathrm{Aff}(\mathbf{Z}/(6))$ has more than 3 elements of order 2, it is isomorphic to D_6 . Since $\mathrm{Aff}(\mathbf{F}_4)$ has more than 2 elements of order 3, it is isomorphic to A_4 .

Group	Order 1	Order 2	Order 3	Order 4	Order 6	Order 12
$\mathbf{Z}/(12)$	1	1	2	2	2	4
$(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$	1	3	2	0	6	0
A_4	1	3	8	0	0	0
D_6	1	7	2	0	2	0
$\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$	1	1	2	6	2	0

TABLE 2. Counting orders of elements in groups of order 12.

In books where groups of order 12 are classified, $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ is often written as T , but it is not given a name matching that label.² Should it be called the “obscure group of order 12”? Actually, this group is in a standard family of finite groups: the dicyclic groups, also called the binary dihedral groups. They are nonabelian with order $4n$ ($n \geq 2$) and each contains a unique element of order 2.³ In $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$, its unique element of order 2 is $(0, 2)$. The dicyclic group of order 8 is Q_8 , and more generally the dicyclic group of order 2^m is the generalized quaternion group Q_{2^m} .

What are the normal subgroups of each group of order 12?

- (1) $\mathbf{Z}/(12)$: each subgroup is normal and there is one of order d for each d dividing 12.
- (2) $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$: each subgroup is normal and is $A \times B$ where A is a subgroup of $(\mathbf{Z}/(2))^2$ and B is a subgroup of $\mathbf{Z}/(3)$. There is one A of order 1, three of order 2, and one of order 4, and there is one B of order 1 and one of order 3.
- (3) A_4 : the normal subgroups are trivial, the unique subgroup of order 4, and A_4 .
- (4) D_6 : the normal subgroups are the subgroups of $\langle r \rangle$ (which are $\langle 1 \rangle$, $\langle r^2 \rangle$, $\langle r^3 \rangle$, and $\langle r \rangle$), two subgroups of order 6 not in $\langle r \rangle$ (which are $\langle r^2, s \rangle$ and $\langle r^2, rs \rangle$), and D_6 .
- (5) $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$: there are unique subgroups of order 1, 2, and 3, and there are three 2-Sylow subgroups, so there are no normal subgroups of order 4. If $H \subset \mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ has order 6 then H is normal since it has index 2, so $g^2 \in H$ for all g . The set of squares in $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ is $\{(0, 0), (1, 0), (2, 0), (0, 2)\}$. Letting $x = (1, 0)$ and $y = (0, 2)$, which are in H , x has order 3 and y has order 2. Check $xy = yx = (1, 2)$, so x and y commute. Thus xy has order 6, so $H = \langle xy \rangle$. Thus there is a unique subgroup of order 6 and it is normal.

Remark 3. In elementary number theory, the sum of the (positive) divisors of an $n \in \mathbf{Z}^+$ is denoted $\sigma(n)$ and n is called perfect when $\sigma(n) = 2n$. The divisors of n are the sizes of the subgroups of $\mathbf{Z}/(n)$. For each finite group G , set $\sigma(G) = \sum_{N \triangleleft G} |N|$, so $\sigma(\mathbf{Z}/(n)) = \sigma(n)$. If $\sigma(G) = 2|G|$, this generalizes perfect numbers and includes them for the special case where G is cyclic. When $\sigma(G) = 2|G|$ we call G a Leinster group, named after Tom Leinster, as “perfect group” already means something else in group theory. Among the groups of order 12, the only one that is Leinster is $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ since $1 + 2 + 3 + 6 + 12 = 24 = 2(12)$. This the smallest nonabelian Leinster group.

We said at the start that Kempe’s list of groups of order 12 has a mistake. Kempe wrote each of his 5 proposed groups in tabular form (as a list of 12 permutations in S_{12}) and called them T_1, T_2, T_3, T_4 , and T_5 . Perhaps Kempe’s T_j -notation is the origin of the notation T

²See [2, pp. 98–99], [4, pp. 178–179, 251–252], and [5, pp. 84–85, 171]. A tetrahedron has 12 orientation-preserving symmetries, but that group of symmetries is isomorphic to A_4 , not to T .

³We can allow $n = 1$, using the cyclic group of order 4, but that is abelian.

for the obscure group of order 12. It turns out that $T_1 \cong \mathbf{Z}/(12)$, $T_2 \cong (\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$, $T_3 \cong D_6$, and $T_5 \cong A_4$, but T_4 , which is shown in Table 3, is not a group and Kempe's list of groups of order 12 did not include one isomorphic to $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$, the obscure group of order 12.

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>i</i>	<i>j</i>	<i>k</i>	<i>l</i>
<i>b</i>	<i>a</i>	<i>d</i>	<i>c</i>	<i>j</i>	<i>i</i>	<i>l</i>	<i>k</i>	<i>f</i>	<i>e</i>	<i>h</i>	<i>g</i>
<i>c</i>	<i>d</i>	<i>b</i>	<i>a</i>	<i>g</i>	<i>h</i>	<i>f</i>	<i>e</i>	<i>k</i>	<i>l</i>	<i>j</i>	<i>i</i>
<i>d</i>	<i>c</i>	<i>a</i>	<i>b</i>	<i>l</i>	<i>k</i>	<i>i</i>	<i>j</i>	<i>h</i>	<i>g</i>	<i>e</i>	<i>f</i>
<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>i</i>	<i>j</i>	<i>k</i>	<i>l</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>f</i>	<i>e</i>	<i>h</i>	<i>g</i>	<i>b</i>	<i>a</i>	<i>d</i>	<i>c</i>	<i>j</i>	<i>i</i>	<i>l</i>	<i>k</i>
<i>g</i>	<i>h</i>	<i>f</i>	<i>e</i>	<i>k</i>	<i>l</i>	<i>j</i>	<i>i</i>	<i>c</i>	<i>d</i>	<i>b</i>	<i>a</i>
<i>h</i>	<i>g</i>	<i>e</i>	<i>f</i>	<i>d</i>	<i>c</i>	<i>a</i>	<i>b</i>	<i>l</i>	<i>k</i>	<i>i</i>	<i>j</i>
<i>i</i>	<i>j</i>	<i>k</i>	<i>l</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>
<i>j</i>	<i>i</i>	<i>l</i>	<i>k</i>	<i>f</i>	<i>e</i>	<i>h</i>	<i>g</i>	<i>b</i>	<i>a</i>	<i>d</i>	<i>c</i>
<i>k</i>	<i>l</i>	<i>j</i>	<i>i</i>	<i>c</i>	<i>d</i>	<i>b</i>	<i>a</i>	<i>g</i>	<i>h</i>	<i>f</i>	<i>e</i>
<i>l</i>	<i>k</i>	<i>i</i>	<i>j</i>	<i>h</i>	<i>g</i>	<i>e</i>	<i>f</i>	<i>d</i>	<i>c</i>	<i>a</i>	<i>b</i>

TABLE 3. Kempe's false group of order 12.

Why are the permutations in Table 3 not a subgroup of S_{12} ? Each row is a permutation of a, b, \dots, k, l and two of the rows are 12-cycles (the 7th and 11th rows) but no row has order 6 (the square of a 12-cycle has order 6) and most rows that have order greater than 2 do not have their inverse in the table (*e.g.*, rows 3 and 4).

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