# GROUPS OF ORDER 12

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The groups of order 12, up to isomorphism, were first determined in the 1880s: Kempe [3, pp. 37–43] gave a list of 5 groups and Cayley [1] pointed out a few years later that one of Kempe's groups did not make sense and that a specific group was missed.

We will use semidirect products to describe all 5 groups of order 12 up to isomorphism. Two are abelian and the others are $A_4$, $D_6$, and a less familiar group.

**Theorem 1.** *Every group of order* 12 *is a semidirect product of a group of order* 3 *and a group of order* 4*.*

*Proof.* Let $|G| = 12 = 2^2 \cdot 3$. A 2-Sylow subgroup has order 4 and a 3-Sylow subgroup has order 3. We will start by showing $G$ has a normal 2-Sylow subgroup or a normal 3-Sylow subgroup: $n_2 = 1$ or $n_3 = 1$. From the Sylow theorems,

$$n_2 \mid 3, \quad n_2 \equiv 1 \bmod 2, \quad n_3 \mid 4, \quad n_3 \equiv 1 \bmod 3.$$

Therefore $n_2 = 1$ or 3 and $n_3 = 1$ or 4.

To show $n_2 = 1$ or $n_3 = 1$, assume $n_3 \neq 1$. Then $n_3 = 4$. Let's count elements of order 3. Since each 3-Sylow subgroup has order 3, different 3-Sylow subgroups intersect trivially. Each of the 3-Sylow subgroups of $G$ contains two elements of order 3, so the number of elements in $G$ of order 3 is $2n_3 = 8$. This leaves us with $12 - 8 = 4$ elements in $G$ not of order 3. A 2-Sylow subgroup has order 4 and contains no elements of order 3, so one 2-Sylow subgroup must account for the remaining 4 elements of $G$. Thus $n_2 = 1$ if $n_3 \neq 1$.

Next we show $G$ is a semidirect product of a 2-Sylow and 3-Sylow subgroup. Let $P_2$ be a 2-Sylow subgroup and $P_3$ be a 3-Sylow subgroup of $G$. Since $P_2$ and $P_3$ have relatively prime orders, $P_2 \cap P_3 = \{1\}$ and the set $P_2P_3 = \{xy : x \in P_2, y \in P_3\}$ has size $|P_2||P_3|/|P_2 \cap P_3| = 12 = |G|$, so $G = P_2P_3$. Since $P_2$ or $P_3$ is normal in $G$, $G$ is a semidirect product of $P_2$ and $P_3$: $G \cong P_2 \rtimes P_3$ if $P_2 \lhd G$ and $G \cong P_3 \rtimes P_2$ if $P_3 \lhd G$.[1] $\square$

Groups of order 4 are isomorphic to $\mathbf{Z}/(4)$ or $(\mathbf{Z}/(2))^2$, and groups of order 3 are isomorphic to $\mathbf{Z}/(3)$, so every group of order 12 is a semidirect product of the form

$$\mathbf{Z}/(4) \rtimes \mathbf{Z}/(3), \quad (\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(3), \quad \mathbf{Z}/(3) \rtimes \mathbf{Z}/(4), \quad \mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2.$$

To determine these up to isomorphism, we work out how $\mathbf{Z}/(4)$ and $(\mathbf{Z}/(2))^2$ act by automorphisms on $\mathbf{Z}/(3)$ and how $\mathbf{Z}/(3)$ acts by automorphisms on $\mathbf{Z}/(4)$ and $(\mathbf{Z}/(2))^2$.

**Theorem 2.** *Every group of order* 12 *is isomorphic to one of* $\mathbf{Z}/(12)$*,* $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$*,* $A_4$*,* $D_6$*, or the nontrivial semidirect product* $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$*.*

We say *the* nontrivial semidirect product $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ since there is only one nontrivial homomorphism $\mathbf{Z}/(4) \to \mathrm{Aut}(\mathbf{Z}/(3)) = (\mathbf{Z}/(3))^\times$, namely $k \bmod 4 \mapsto (-1)^k \bmod 3$. The

[1] The notation $P_2 \rtimes P_3$ could refer to more than one group since there could be different actions $P_3 \to \mathrm{Aut}(P_2)$ leading to nonisomorphic semidirect products.

corresponding group $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ has group law

$$(a,b)(c,d) = (a + (-1)^b c, b+d). \tag{1}$$

This is generated by $x = (1,0)$ and $y = (0,1)$ with $x^3 = 1$, $y^4 = 1$, and $yxy^{-1} = x^{-1}$. A model for this group inside $\mathrm{SL}_2(\mathbf{C})$ has $x = \left(\begin{smallmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{smallmatrix}\right)$ with $\omega = e^{2\pi i/3}$ and $y = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$.

In the proof of Theorem 2, we will appeal to an isomorphism property of semidirect products: for each semidirect product $H \rtimes_\varphi K$ and automorphism $f\colon K \to K$, $H \rtimes_\varphi K \cong H \rtimes_{\varphi \circ f} K$. This says that precomposing an action of $K$ on $H$ by automorphisms (that's $\varphi$) with an automorphism of $K$ produces an isomorphic semidirect product of $H$ and $K$.

*Proof.* Here are automorphisms of possible Sylow subgroups: $\mathrm{Aut}(\mathbf{Z}/(4)) \cong (\mathbf{Z}/(4))^\times = \{\pm 1 \bmod 4\}$, $\mathrm{Aut}((\mathbf{Z}/(2))^2) \cong \mathrm{GL}_2(\mathbf{Z}/(2))$, and $\mathrm{Aut}(\mathbf{Z}/(3)) \cong (\mathbf{Z}/(3))^\times = \{\pm 1 \bmod 3\}$.

Case 1: $n_2 = 1$, $P_2 \cong \mathbf{Z}/(4)$.

The 2-Sylow subgroup is normal, so the 3-Sylow subgroup acts on it. Our group is a semidirect product $\mathbf{Z}/(4) \rtimes \mathbf{Z}/(3)$, for which the action of the second group on the first is through a homomorphism $\varphi\colon \mathbf{Z}/(3) \to (\mathbf{Z}/(4))^\times$. The domain has order 3 and the target has order 2, so this homomorphism is trivial, and thus the semidirect product must be trivial: it's the direct product

$$\mathbf{Z}/(4) \times \mathbf{Z}/(3),$$

which is cyclic of order 12 (generator $(1,1)$).

Case 2: $n_2 = 1$, $P_2 \cong \mathbf{Z}/(2) \times \mathbf{Z}/(2)$.

We need to understand all homomorphisms $\varphi\colon \mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$. The trivial homomorphism leads to the direct product

$$(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3).$$

What about nontrivial homomorphisms $\varphi\colon \mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$? Inside $\mathrm{GL}_2(\mathbf{Z}/(2))$, which has order 6 (it's isomorphic to $S_3$), there is one subgroup of order 3: $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}$. A nontrivial homomorphism $\varphi\colon \mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$ is determined by where it sends $1 \bmod 3$, which must go to a solution of $A^3 = I_2$; then $\varphi(k \bmod 3) = A^k$ in general. For $\varphi$ to be nontrivial, $A$ needs to have order 3, and there are two choices for that. The two matrices of order 3 in $\mathrm{GL}_2(\mathbf{Z}/(2))$ are inverses. Call one of them $A$, making the other $A^{-1}$. The resulting homomorphisms $\mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$ are $\varphi(k \bmod 3) = A^k$ and $\psi(k \bmod 3) = A^{-k}$, which are related to each other by composition with inversion, but *watch out*: inversion is not an automorphism of $\mathrm{GL}_2(\mathbf{Z}/(2))$. It is an automorphism of $\mathbf{Z}/(3)$, where it's negation. So precomposing $\varphi$ with negation on $\mathbf{Z}/(3)$ turns $\varphi$ into $\psi$: $\psi = \varphi \circ f$, where $f(x) = -x$ on $\mathbf{Z}/(3)$. Therefore the two nontrivial homomorphisms $\mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$ are linked through precomposition with an automorphism of $\mathbf{Z}/(3)$, so $\varphi$ and $\psi$ define isomorphic semidirect products. Thus up to isomorphism, there is one nontrivial semidirect product

$$(\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(3).$$

Since up to isomorphism one group of order 12 has $n_2 = 1$ and a noncyclic 2-Sylow subgroup, and $A_4$ also fits this description, this semidirect product is isomorphic to $A_4$.

Now assume $n_2 \neq 1$, so $n_2 = 3$ and $n_3 = 1$. Since $n_2 > 1$, the group is nonabelian, so it's a nontrivial semidirect product (a direct product of abelian groups is abelian).

Case 3: $n_2 = 3$, $n_3 = 1$, and $P_2 \cong \mathbf{Z}/(4)$.

Our group looks like $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$, built from a nontrivial homomorphism $\varphi\colon \mathbf{Z}/(4) \to \mathrm{Aut}(\mathbf{Z}/(3)) = (\mathbf{Z}/(3))^\times$ There is only one choice of $\varphi$: it has to send $1 \bmod 4$ to $-1 \bmod$

3, which determines everything else: $\varphi(c \bmod 4) = (-1)^c \bmod 3$. Therefore there is one nontrivial semidirect product $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ and its group operation is given by (1).

Case 4: $n_2 = 3$, $n_3 = 1$, and $P_2 \cong \mathbf{Z}/(2) \times \mathbf{Z}/(2)$.

The group is $\mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2$ for a nontrivial homomorphism $\varphi\colon (\mathbf{Z}/(2))^2 \to (\mathbf{Z}/(3))^\times$. The group $(\mathbf{Z}/(2))^2$ has a pair of generators $(1,0)$ and $(0,1)$, and $\varphi(a,b) = \varphi(1,0)^a\varphi(0,1)^b$, where $\varphi(1,0)$ and $\varphi(0,1)$ are $\pm 1$. Conversely, this formula for $\varphi$ defines a homomorphism since $a$ and $b$ are in $\mathbf{Z}/(2)$ and exponents on $\pm 1$ only matter mod 2. For $\varphi$ to be nontrivial means $\varphi(1,0)$ and $\varphi(0,1)$ are not both 1, so there are three choices of $\varphi\colon (\mathbf{Z}/(2))^2 \to (\mathbf{Z}/(3))^\times$:

$$\varphi(a,b) = (-1)^a, \quad \varphi(a,b) = (-1)^b, \quad \varphi(a,b) = (-1)^a(-1)^b = (-1)^{a+b}.$$

This does *not* mean the three corresponding semidirect products $\mathbf{Z}/(3) \rtimes_\varphi (\mathbf{Z}/(2))^2$ are nonisomorphic. In fact, the above three choices of $\varphi$ lead to isomorphic semidirect products: precomposing the first $\varphi$ with the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ produces the second $\varphi$, and precomposing the first $\varphi$ with the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ produces the third $\varphi$. Therefore the three nontrivial semidirect products $\mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2$ are isomorphic, so all groups of order 12 with $n_2 = 3$ (equivalently, all nonabelian groups of order 12 with $n_3 = 1$) and 2-Sylow subgroup isomorphic to $(\mathbf{Z}/(2))^2$ are isomorphic. One such group is $D_6$, with normal 3-Sylow subgroup $\{1, r^2, r^4\}$. $\square$

For a group of order 12, Table 1 lists structural properties to know it up to isomorphism. (That $n_3 = 4$ implies $G \cong A_4$ is because $G$ acting by conjugation on its 4 3-Sylow subgroups is an isomorphism of $G$ with $A_4$.)

| Group | Abelian? | $n_2$ | $n_3$ | 2-Sylow |
|---|---|---|---|---|
| $\mathbf{Z}/(12)$ | Yes | 1 | 1 | cyclic |
| $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$ | Yes | 1 | 1 | noncyclic |
| $A_4$ | No | 1 | 4 | noncyclic |
| $D_6$ | No | 3 | 1 | noncyclic |
| $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ | No | 3 | 1 | cyclic |

Table 1. Structural properties of groups of order 12.

For example, here are five groups of order 12:

$$\mathbf{Z}/(2) \times \mathbf{Z}/(6), \quad \mathbf{Z}/(2) \times S_3, \quad \mathrm{PSL}_2(\mathbf{F}_3), \quad \mathrm{Aff}(\mathbf{Z}/(6)), \quad \mathrm{Aff}(\mathbf{F}_4). \tag{2}$$

The first group is abelian with noncyclic 2-Sylow subgroup, so it's isomorphic to $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$ (or use the Chinese remainder theorem). The remaining groups are nonabelian. Since $\mathbf{Z}/(2) \times S_3$ has $n_3 = 1$ and a noncyclic 2-Sylow subgroup, $\mathbf{Z}/(2) \times S_3 \cong D_6$. The group $\mathrm{PSL}_2(\mathbf{F}_3)$ has $n_3 > 1$, so $\mathrm{PSL}_2(\mathbf{F}_3) \cong A_4$. The group $\mathrm{Aff}(\mathbf{Z}/(6))$ has $n_2 > 1$ and noncyclic 2-Sylow subgroup, so $\mathrm{Aff}(\mathbf{Z}/(6)) \cong D_6$. Finally, $\mathrm{Aff}(\mathbf{F}_4)$ has $n_3 > 1$, so $\mathrm{Aff}(\mathbf{F}_4) \cong A_4$.

Another way to distinguish between groups of order 12 is by counting elements of a certain order. From Table 2 below, these groups can be distinguished by counting elements of order 2 except for $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(4)$ and $A_4$, where one is abelian and the other isn't.

For example, among the groups in (2), $\mathbf{Z}/(2) \times \mathbf{Z}/(6)$ is abelian with three elements of order 2, so it is isomorphic to $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$. Since $\mathbf{Z}/(2) \times S_3$ has more than 3 elements of order 2, it is isomorphic to $D_6$. Since $\mathrm{PSL}_2(\mathbf{F}_3)$ has more than 2 elements of order 3, it is isomorphic to $A_4$. Since $\mathrm{Aff}(\mathbf{Z}/(6))$ has more than 3 elements of order 2, it is isomorphic to $D_6$. Since $\mathrm{Aff}(\mathbf{F}_4)$ has has more than 2 elements of order 3, it is isomorphic to $A_4$.

| Group | Order 1 | Order 2 | Order 3 | Order 4 | Order 6 | Order 12 |
|---|---|---|---|---|---|---|
| $\mathbf{Z}/(12)$ | 1 | 1 | 2 | 2 | 2 | 4 |
| $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$ | 1 | 3 | 2 | 0 | 6 | 0 |
| $A_4$ | 1 | 3 | 8 | 0 | 0 | 0 |
| $D_6$ | 1 | 7 | 2 | 0 | 2 | 0 |
| $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ | 1 | 1 | 2 | 6 | 2 | 0 |

TABLE 2. Counting orders of elements in groups of order 12.

In books where groups of order 12 are classified, $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ is often written as $T$, but it is not given a name matching that label.[2] Should it be called the "obscure group of order 12"? Actually, this group is in a standard family of finite groups: the dicyclic groups, also called the binary dihedral groups. They are nonabelian with order $4n$ ($n \geq 2$) and each contains a unique element of order 2.[3] In $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$, its unique element of order 2 is $(0, 2)$. The dicyclic group of order 8 is $Q_8$, and more generally the dicyclic group of order $2^m$ is the generalized quaternion group $Q_{2^m}$.

We said at the start that Kempe's list of groups of order 12 has a mistake. Kempe wrote each of his 5 proposed groups in tabular form (as a list of 12 permutations in $S_{12}$) and called them $T_1$, $T_2$, $T_3$, $T_4$, and $T_5$. It turns out that $T_1 \cong \mathbf{Z}/(12)$, $T_2 \cong (\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$, $T_3 \cong D_6$, and $T_5 \cong A_4$, but $T_4$, which is shown in Table 3, is not a group and Kempe's list of groups of order 12 did not include a group isomorphic to $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ | $i$ | $j$ | $k$ | $l$ |
| $b$ | $a$ | $d$ | $c$ | $j$ | $i$ | $l$ | $k$ | $f$ | $e$ | $h$ | $g$ |
| $c$ | $d$ | $b$ | $a$ | $g$ | $h$ | $f$ | $e$ | $k$ | $l$ | $j$ | $i$ |
| $d$ | $c$ | $a$ | $b$ | $l$ | $k$ | $i$ | $j$ | $h$ | $g$ | $e$ | $f$ |
| $e$ | $f$ | $g$ | $h$ | $i$ | $j$ | $k$ | $l$ | $a$ | $b$ | $c$ | $d$ |
| $f$ | $e$ | $h$ | $g$ | $b$ | $a$ | $d$ | $c$ | $j$ | $i$ | $l$ | $k$ |
| $g$ | $h$ | $f$ | $e$ | $k$ | $l$ | $j$ | $i$ | $c$ | $d$ | $b$ | $a$ |
| $h$ | $g$ | $e$ | $f$ | $d$ | $c$ | $a$ | $b$ | $l$ | $k$ | $i$ | $j$ |
| $i$ | $j$ | $k$ | $l$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
| $j$ | $i$ | $l$ | $k$ | $f$ | $e$ | $h$ | $g$ | $b$ | $a$ | $d$ | $c$ |
| $k$ | $l$ | $j$ | $i$ | $c$ | $d$ | $b$ | $a$ | $g$ | $h$ | $f$ | $e$ |
| $l$ | $k$ | $i$ | $j$ | $h$ | $g$ | $e$ | $f$ | $d$ | $c$ | $a$ | $b$ |

TABLE 3. Kempe's false group of order 12.

Why are the permutations in Table 3 not a subgroup of $S_{12}$? Each row is a permutation of $a, b, \ldots, k, l$ and two of the rows are 12-cycles (the 7th and 11th rows) but no row has order 6 (the square of a 12-cycle has order 6) and most rows that have order greater than 2 do not have their inverse in the table (*e.g.*, rows 3 and 4). Perhaps Kempe's $T_j$-notation is the origin of the notation $T$ for the obscure group of order 12.

[2] See [2, pp. 98–99], [4, pp. 178–179, 251-252], and [5, pp. 84–85, 171]. A tetrahedron has 12 orientation-preserving symmetries, but that group of symmetries is isomorphic to $A_4$, not to $T$.

[3] We can allow $n = 1$, using the cyclic group of order 4, but that is abelian.

## References

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