NORMAL SUBGROUPS OF $\mathrm{Aff}(F)$

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For each field F we will find all the normal subgroups of

$$\mathrm{Aff}(F) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in F^\times, b \in F \right\}.$$

The group law in $\mathrm{Aff}(F)$ is

$$(1) \qquad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa' & ab'+b \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix}.$$

By (1), projection $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mapsto a$ is a homomorphism from $\mathrm{Aff}(F)$ to $F^\times$, so for each subgroup H of $F^\times$ we have the subgroup

$$(2) \qquad \begin{pmatrix} H & F \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in H, b \in F \right\}$$

of $\mathrm{Aff}(F)$. Since $H \lhd F^\times$ (as $F^\times$ is abelian), the group (2) is a normal subgroup of $\mathrm{Aff}(F)$: the inverse image of a normal subgroup under a surjective group homomorphism is a normal sugbroup. Denote the group (2) as N_H.

Theorem 1. *Every nontrivial normal subgroup of* $\mathrm{Aff}(F)$ *is* N_H *for a subgroup* H *of* $F^\times$.

Proof. Let N be a nontrivial normal subgroup of $\mathrm{Aff}(F)$. It is a union of conjugacy classes and larger than the identity, so let's see what the conjugacy class of each non-identity matrix is. Conjugation in $\mathrm{Aff}(F)$ is

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & (1-a)y+bx \\ 0 & 1 \end{pmatrix},$$

and if $a \neq 1$ then $\{(1-a)y + b : y \in F\} = F$. Thus conjugating by matrices with $x = 1$ gives us

$$(3) \qquad a \neq 1 \Longrightarrow \text{conjugacy class of } \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \text{ is } \begin{pmatrix} a & F \\ 0 & 1 \end{pmatrix}.$$

Setting $a = 1$,

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & bx \\ 0 & 1 \end{pmatrix},$$

so if $b \neq 0$ and we let x run over $F^\times$ we get

$$(4) \qquad b \neq 0 \Longrightarrow \text{conjugacy class of } \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ is } \begin{pmatrix} 1 & F^\times \\ 0 & 1 \end{pmatrix}.$$

To prove $N = N_H$ for some subgroup H of $F^\times$, observe that if $N = N_H$ then H is the set of upper-left entries in the matrices of N. Therefore *define*

$$H = \left\{ a \in F^\times : \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in N \text{ for some } b \in F \right\}.$$

Then H is a subgroup of $F^\times$: $1 \in H$ because $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) \in N$ and (1) shows that if $a, a' \in H$ then aa' and $1/a$ are also in H. We will prove $N = N_H$.

Showing the containment $N \subset N_H$ is just a matter of unwinding definitions: if $\left(\begin{smallmatrix}a & b\\ 0 & 1\end{smallmatrix}\right) \in N$ then $a \in H$, so $\left(\begin{smallmatrix}a & b\\ 0 & 1\end{smallmatrix}\right) \in N_H$ by the meaning of N_H. Thus $N \subset N_H$.

To prove $N_H \subset N$ we will show $a \in H \Longrightarrow \left(\begin{smallmatrix}a & F\\ 0 & 1\end{smallmatrix}\right) \subset N$. Then if $\left(\begin{smallmatrix}a & b\\ 0 & 1\end{smallmatrix}\right) \in N_H$ we have $a \in H$, so $\left(\begin{smallmatrix}a & b\\ 0 & 1\end{smallmatrix}\right) \in \left(\begin{smallmatrix}a & F\\ 0 & 1\end{smallmatrix}\right) \subset N$, and thus $N_H \subset N$.

Case 1: $H \neq \{1\}$.

Pick $a \in H - \{1\}$. To show $\left(\begin{smallmatrix}a & F\\ 0 & 1\end{smallmatrix}\right) \subset N$, by the definition of H we have $\left(\begin{smallmatrix}a & b\\ 0 & 1\end{smallmatrix}\right) \in N$ for some $b \in F$. The conjugacy class of $\left(\begin{smallmatrix}a & b\\ 0 & 1\end{smallmatrix}\right)$ is in N by normality, so $\left(\begin{smallmatrix}a & F\\ 0 & 1\end{smallmatrix}\right) \subset N$ by (3).

Next we show $\left(\begin{smallmatrix}1 & F\\ 0 & 1\end{smallmatrix}\right) \subset N$. Pick an $a \neq 1$ in H, so $\left(\begin{smallmatrix}a & F\\ 0 & 1\end{smallmatrix}\right) \subset N$. Since $1/a \in H$ and $1/a \neq 1$, $\left(\begin{smallmatrix}1/a & F\\ 0 & 1\end{smallmatrix}\right) \subset N$ too. In particular, N contains $\left(\begin{smallmatrix}a & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1/a & 1/a\\ 0 & 1\end{smallmatrix}\right)$, so N contains the product

$$\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1/a & 1/a\\ 0 & 1\end{pmatrix} = \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}.$$

The conjugacy class of $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ is $\left(\begin{smallmatrix}1 & F^\times\\ 0 & 1\end{smallmatrix}\right)$ by (4), so $\left(\begin{smallmatrix}1 & F^\times\\ 0 & 1\end{smallmatrix}\right) \subset N$. The identity $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ is in N too, so $\left(\begin{smallmatrix}1 & F\\ 0 & 1\end{smallmatrix}\right) \subset N$.

Case 2: $H = \{1\}$.

In this case, $N \subset N_{\{1\}} = \left(\begin{smallmatrix}1 & F\\ 0 & 1\end{smallmatrix}\right)$. To prove the reverse containment, since N is nontrivial there is some $\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)$ in N with $b \in F^\times$. Then the conjugacy class of $\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)$ is in N, so $\left(\begin{smallmatrix}1 & F^\times\\ 0 & 1\end{smallmatrix}\right) \subset N$. Also the identity $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ is in N, so N contains $\left(\begin{smallmatrix}1 & F\\ 0 & 1\end{smallmatrix}\right) = N_{\{1\}}$.

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