1. Introduction

The group $\text{SL}_2(\mathbb{R})$ is not easy to visualize: it naturally lies in $M_2(\mathbb{R})$, which is 4-dimensional (the entries of a variable $2 \times 2$ real matrix are 4 free parameters). We will derive a product decomposition for $\text{SL}_2(\mathbb{R})$ and use it to get a concrete image of $\text{SL}_2(\mathbb{R})$.

Inside $\text{SL}_2(\mathbb{R})$ are the following three subgroups:

$K = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \right\}$, $A = \left\{ \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} : r > 0 \right\}$, $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}$.

Theorem 1.1. We have a decomposition $\text{SL}_2(\mathbb{R}) = KAN$: every $g \in \text{SL}_2(\mathbb{R})$ has a unique representation as $g = kan$ where $k \in K$, $a \in A$, and $n \in N$.

This formula $\text{SL}_2(\mathbb{R}) = KAN$ is called the Iwasawa decomposition of the group. Don’t confuse the use of $a$ in Theorem 1.1 as the label for a matrix in $A$ with $a$ as a real number in the matrix $(a \ b \ c \ d)$. The distinction should always be clear from the context. Since $\text{SL}_2(\mathbb{R})$ is defined by the single equation $ad - bc = 1$ inside of $M_2(\mathbb{R})$, it is a manifold of dimension $4 - 1 = 3$. The subgroups $K$, $A$, and $N$ are each 1-dimensional ($K \cong S^1$, $A \cong \mathbb{R}_{>0}$, and $N \cong \mathbb{R}$), and Theorem 1.1 shows they fully account for the 3 dimensions of $\text{SL}_2(\mathbb{R})$.

The subgroups in the Iwasawa decomposition are related to conjugacy classes. We will see that a matrix in $\text{SL}_2(\mathbb{R})$ is, up to sign, conjugate to a matrix in $K$, $A$, or $N$.

2. Iwasawa Decomposition

To derive the Iwasawa decomposition of $\text{SL}_2(\mathbb{R})$ we will use an action of this group on bases in $\mathbb{R}^2$.

For $g = (a \ b \ c \ d)$ in $\text{SL}_2(\mathbb{R})$, apply it to the standard basis $e_1, e_2$. The vectors

$ge_1 = \begin{pmatrix} a \\ c \end{pmatrix}$, $ge_2 = \begin{pmatrix} b \\ d \end{pmatrix}$

are also a basis of $\mathbb{R}^2$. We will pass from this new basis of $\mathbb{R}^2$ back to the standard basis $e_1, e_2$ of $\mathbb{R}^2$ by a sequence of transformations in $\text{SL}_2(\mathbb{R})$ that amounts to something like the Gram–Schmidt process (which turns a basis of $\mathbb{R}^n$ into an orthonormal basis of $\mathbb{R}^n$).

Let $\theta$ be the angle from the positive $x$-axis to $ge_1$. Let $\rho_0$ be the counterclockwise rotation of the plane around the origin by $\theta$, so $\rho_{-\theta}(ge_1)$ is on the positive $x$-axis. Because $\det g$ is positive, the ordered pair of vectors $(ge_1, ge_2)$ has the same orientation as the ordered pair $(e_1, e_2)$, so $\rho_{-\theta}(ge_2)$ is in the upper (rather than lower) half-plane.

Since $\rho_{-\theta}(ge_1)$ is a positive scalar multiple of $e_1$, we want to divide $\rho_{-\theta}(ge_1)$ by its length so it becomes $e_1$. Its length is $r = ||\rho_{-\theta}(ge_1)|| = ||ge_1|| = \sqrt{a^2 + c^2}$. Applying $\begin{pmatrix} 1/r & 0 \\ 0 & 1/r \end{pmatrix}$ will have the desired effect $\rho_{-\theta}(ge_1) \mapsto e_1$, but this matrix doesn’t have determinant 1. On the other hand, $\begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix}$ also has the desired effect on $\rho_{-\theta}(ge_1)$ and has determinant 1. So
apply the matrix \((1/r \ 0\ 0\ r)\) to \(\mathbb{R}^2\). It sends \(\rho\_\theta(ge_1)\) to \((1/r \ 0\ 0\ r)\rho\_\theta(ge_1) = e_1\). What does it do to \(\rho\_\theta(ge_2)\)? The vector \((1/r \ 0\ 0\ r)\rho\_\theta(ge_2)\) is in the upper half-plane (because \((1/r \ 0\ 0\ r)\) has positive determinant) and along with \(e_1\) it forms two edges of a parallelogram with area 1 (because \((1/r \ 0\ 0\ r)\) has determinant \(\pm 1\)). A parallelogram with area 1 having base \(e_1\) must have height 1, so \((1/r \ 0\ 0\ r)\rho\_\theta(ge_2) = (\hat{z})\) for some \(x\).

Each horizontal shear transformation \((1 \ 1\ 0\ -x)\), which has determinant 1, fixes the \(x\)-axis and acts as a stretching along each horizontal line. Applying the horizontal shear transformation \((1 \ 1\ 0\ -x)\) to \(\mathbb{R}^2\) takes \((\hat{z})\) to \((0\ 1) = e_2\) and fixes \(e_1\). We have finally returned to the standard basis \(e_1, e_2\) from the basis \(ge_1, ge_2\) by a sequence of transformations in \(\text{SL}_2(\mathbb{R})\). Our overall composite transformation is

\[
\begin{pmatrix}
1 & -x \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1/r & 0 \\
0 & r
\end{pmatrix}
\rho\_\theta,
\]

so

\[
\begin{pmatrix}
1 & -x \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1/r & 0 \\
0 & r
\end{pmatrix}
\rho\_\theta g
\]

sends \(e_1\) to \(e_1\) and \(e_2\) to \(e_2\). A linear transformation on \(\mathbb{R}^2\) is determined by what it does to a basis, so

\[
\begin{pmatrix}
1 & -x \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1/r & 0 \\
0 & r
\end{pmatrix}
\rho\_\theta g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Solving for \(g\),

\[
g = \rho\_\theta \left( \frac{1}{r} \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}
\]

\(\in\ KAN.\)

Such an expression for \(g\) as a product \(kan\) with \(k \in K, a \in A,\) and \(n \in N\) is called the Iwasawa decomposition of \(g\).

To check this decomposition is unique, for each angle \(\theta, r > 0,\) and \(x \in \mathbb{R},\) set

\[
g = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r \cos \theta & xr \cos \theta - (1/r) \sin \theta \\ r \sin \theta & xr \sin \theta + (1/r) \cos \theta \end{pmatrix}.
\]

If this is \((a \ b \ c \ d) \in \text{SL}_2(\mathbb{R})\) then

\[(2.1) \quad r = \sqrt{a^2 + c^2} > 0, \quad \cos \theta = \frac{a}{r}, \quad \sin \theta = \frac{c}{r},\]

and

\[
x = \begin{pmatrix} b + (1/r) \sin \theta \\ r \cos \theta \end{pmatrix}, \quad \text{if} \ \cos \theta \neq 0,
\]

\[
x = \begin{pmatrix} a \cos \theta \sin \theta \\ d - (1/r) \cos \theta \end{pmatrix}, \quad \text{if} \ \sin \theta \neq 0.
\]

Substituting the formulas for \(\cos \theta\) and \(\sin \theta\) into the formula for \(x,\) and using \(ad - bc = 1,\) we obtain the uniform formula

\[(2.2) \quad x = \frac{ab + cd}{a^2 + c^2}.
\]

All the parameters in the matrices making up the Iwasawa decomposition of \((a \ b \ c \ d)\) are determined in \((2.1)\) and \((2.2),\) so the Iwasawa decomposition is unique. This completes
the proof of Theorem 1.1. In an appendix we derive the Iwasawa decomposition using a different action of $\text{SL}_2(\mathbb{R})$, on the upper half-plane.

The Iwasawa decomposition for $\text{SL}_2(\mathbb{R})$ extends to higher dimensions: $\text{SL}_n(\mathbb{R}) = KAN$ where $K = \text{SO}_n(\mathbb{R}) = \{ T \in \text{GL}_n(\mathbb{R}) : TT^T = I_n, \det T = 1 \}$, $A$ is the group of diagonal matrices with positive diagonal entries (and determinant 1) and $N$ is the group of upper-triangular matrices with 1’s along the main diagonal. While $A$ and $N$ are both isomorphic to $\mathbb{R}$ when $n = 2$, $N$ becomes nonabelian for $n > 2$. The group $K$ is compact, the group $A \cong (\mathbb{R}_{>0})^{n-1} \cong \mathbb{R}^{n-1}$ is abelian, and $N$ is a nilpotent group. This explains the notation $A$ and $N$, for abelian and nilpotent.

Returning to the case of $2 \times 2$ matrices, since

\[
(2.3) \quad \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & r^2x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix},
\]

we can move each element of $A$ past an element of $N$ (on either side) at the cost of changing the element of $N$. Therefore $AN = NA$ is a subgroup of $\text{SL}_2(\mathbb{R})$. Explicitly,

\[
(2.4) \quad AN = \left\{ \begin{pmatrix} y & x \\ 0 & 1/y \end{pmatrix} : y > 0, x \in \mathbb{R} \right\}.
\]

The Iwasawa decomposition $KAN = K(AN)$ for $\text{SL}_2(\mathbb{R})$ is the analogue of the polar decomposition $S^1 \times \mathbb{R}_{>0}$ for $\mathbb{C}^\times$.

In the Iwasawa decomposition, neither $K$ nor $AN$ (nor $A$ or $N$) is normal in $\text{SL}_2(\mathbb{R})$. For example, the conjugate of an element of $K$ by $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is usually not in $K$ and the conjugate of an element of $AN$ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is usually not in $AN$. Because of the non-normality, it is not easy to describe the group operation in $\text{SL}_2(\mathbb{R})$ in terms of its Iwasawa decomposition. This decomposition is important for other purposes, such as the following.

**Corollary 2.1.** As a topological space, $\text{SL}_2(\mathbb{R})$ is homeomorphic to the inside of a solid torus.

**Proof.** Let $f : K \times A \times N \to \text{SL}_2(\mathbb{R})$ by $f(k, a, n) = kan$. This is continuous, and by Theorem 1.1 it is surjective. We can write down an inverse function using the computations at the end of the proof of Theorem 1.1. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\text{SL}_2(\mathbb{R})$, define $r(g) = \sqrt{a^2 + c^2}$ and

\[
k(g) = \begin{pmatrix} a/r(g) & -c/r(g) \\ c/r(g) & a/r(g) \end{pmatrix}, \quad a(g) = \begin{pmatrix} r(g) & 0 \\ 0 & 1/r(g) \end{pmatrix},
\]

\[
n(g) = \begin{pmatrix} 1 & (ab + cd)/(a^2 + c^2) \\ 0 & 1 \end{pmatrix}.
\]

The function $g \mapsto (k(g), a(g), n(g))$ from $\text{SL}_2(\mathbb{R})$ to $K \times A \times N$ is continuous and is an inverse to $f$.

Topologically, $K \cong S^1$, $A \cong \mathbb{R}_{>0} \cong \mathbb{R}$, and $N \cong \mathbb{R}$. Therefore topologically, $\text{SL}_2(\mathbb{R}) \cong S^1 \times \mathbb{R}^2$. The plane $\mathbb{R}^2$ is homeomorphic to the open unit disc $D$ by $v \mapsto v/\sqrt{1 + ||v||^2}$

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1 The word nilpotent has different meanings in group theory and matrix theory. A group $G$ is called nilpotent if there is a finite tower of subgroups $\{e\} = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_r = G$ for some $r$ where $N_i \lhd N$ and $N_{i+1}/N_i \subset Z(G/N_i)$ for all $i$. That the subgroup $N$ of $\text{GL}_n(\mathbb{R})$ is nilpotent is shown on pages 27 and 28 of [https://kconrad.math.uconn.edu/blurbs/grouptheory/subgpseries1.pdf](https://kconrad.math.uconn.edu/blurbs/grouptheory/subgpseries1.pdf), where $N$ is written as $\text{UT}_n(\mathbb{R})$. A square matrix is called nilpotent if it has a power equal to $O$, so invertible square matrices are never nilpotent in the matrix sense. Thus a subgroup of $\text{GL}_n(\mathbb{R})$ might be nilpotent in the sense of group theory, but its elements are not nilpotent in the sense of matrix theory.
of Corollary 2.1 gives us the decomposition of a triangular matrix 

\[
\begin{pmatrix}
1 & 0 \\
0 & y
\end{pmatrix}
\]  

for each positive integer \(d\). As an alternate argument, since \(\pi_1(SL(2,\mathbb{R}))\) is isomorphic to \(\pi_1(S^1)\) \(\cong\mathbb{Z}\). From the connection between covering spaces and subgroups of the fundamental group, \(SL(2,\mathbb{R})\) admits a unique covering space of degree \(d\) for each positive integer \(d\) and the universal covering space of \(SL(2,\mathbb{R})\) is the inside of a solid cylinder \(\mathbb{R} \times D\) (homeomorphic to \(\mathbb{R}^3\)). The degree-2 cover of \(SL(2,\mathbb{R})\) is an important group called the metaplectic group.

We can write down an explicit example of a noncontractible loop in \(SL(2,\mathbb{R})\): the subgroup \(K\), or rather the obvious map \(\mathbb{R}\) to \(K\). To prove this loop is noncontractible in \(SL(2,\mathbb{R})\) we use the Iwasawa decomposition to write down a strong deformation retract from \(SL(2,\mathbb{R})\) to \(K\). Let \(h: SL(2,\mathbb{R}) \times [0,1] \to K\) by

\[
h(kan,t) = k\alpha^t\alpha^t = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & r^t
\end{pmatrix}
\begin{pmatrix}
1 & tx \\
0 & 1
\end{pmatrix}.
\]

This is continuous with \(h(kan,0) = k\), \(h(kan,1) = kan\), and \(h(k,t) = k\). Therefore \(\pi_1(SL(2,\mathbb{R})) \cong \pi_1(K) \cong \mathbb{Z}\), so \(K\) has to be a noncontractible loop in \(SL(2,\mathbb{R})\) since \(K\) is noncontractible in \(K\).

That the Iwasawa decomposition gives us a picture of \(SL(2,\mathbb{R})\) is a striking geometric application. Here is an algebraic application (whose punchline is the corollary).

**Theorem 2.3.** The only continuous homomorphism \(SL(2,\mathbb{R}) \to \mathbb{R}\) is the trivial homomorphism.

**Proof.** Let \(f: SL(2,\mathbb{R}) \to \mathbb{R}\) be a continuous homomorphism. Then

\[
f(kan) = f(k) + f(a) + f(n).
\]

We will show \(f\) is trivial on \(K\), \(A\), and \(N\), and thus \(f\) is trivial on \(KAN = SL(2,\mathbb{R})\).

Since \(K \cong S^1\), the elements of finite order in \(K\) are dense. Since \(\mathbb{R}\) has no elements of finite order except 0, \(f\) is trivial on a dense subset of \(K\) and thus is trivial on \(K\) by continuity. (As an alternate argument, since \(K\) is a compact group so is \(f(K)\), and the only compact subgroup of \(\mathbb{R}\) is \(\{0\}\).)

Now we look at \(f\) on \(A\) and \(N\). Since \(A \cong \mathbb{R}_{>0} \cong \mathbb{R}\) by \((r,0) \mapsto \log r\) and \(N \cong \mathbb{R}\) by \((0,t) \mapsto x\), both algebraically and topologically, describing the continuous homomorphisms from \(A\) and \(N\) to \(\mathbb{R}\) is the same as describing the continuous homomorphisms from \(\mathbb{R}\) to
All continuous homomorphisms $\mathbb{R} \to \mathbb{R}$ have the form $x \mapsto tx$ for some real number $t$ (see where 1 goes, call that $t$, and then appeal to the denseness of $\mathbb{Q}$ in $\mathbb{R}$). Therefore

$$f \left( \begin{array}{cc} r & 0 \\ 0 & 1/r \end{array} \right) = t \log r, \quad f \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) = t'x$$

for some $t$ and $t'$. Applying $f$ to both sides of (2.3),

$$t \log r + t'x = t'r^2x + t \log r,$$

so $t'x = t'r^2x$ for all $r > 0$ and $x \in \mathbb{R}$. Thus $t' = 0$ (e.g., take $x = 1$ and $r = 2$ to see this.) This shows $f$ is trivial on $N$.

It remains to show $f$ is trivial on $A$. For this we appeal to the conjugation relation

$$\left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} r & 0 \\ 0 & 1/r \end{array} \right) \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)^{-1} = \left( \begin{array}{cc} 1/r & 0 \\ 0 & r \end{array} \right)^{-1} = \left( \begin{array}{cc} r & 0 \\ 0 & 1/r \end{array} \right)^{-1}.$$

Applying $f$, we get $f(\left( \begin{array}{cc} r & 0 \\ 0 & 1/r \end{array} \right)) = -f(\left( \begin{array}{cc} r & 0 \\ 0 & 1/r \end{array} \right))$, so $f(\left( \begin{array}{cc} r & 0 \\ 0 & 1/r \end{array} \right)) = 0$. \qed

**Corollary 2.4.** Every continuous homomorphism $\text{SL}_2(\mathbb{R}) \to \text{GL}_n(\mathbb{R})$ has image in $\text{SL}_n(\mathbb{R})$.

**Proof.** Let $f : \text{SL}_2(\mathbb{R}) \to \text{GL}_n(\mathbb{R})$ be a continuous homomorphism. Composing $f$ with the determinant $\text{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$ gives a continuous homomorphism $\det \circ f : \text{SL}_2(\mathbb{R}) \to \mathbb{R}^\times$. Since $\text{SL}_2(\mathbb{R})$ is connected (Corollary 2.1), its image under $\det \circ f$ is a connected subgroup of $\mathbb{R}^\times$, so it lies in $\mathbb{R}_{>0}$. As $\mathbb{R}_{>0} \cong \mathbb{R}$ both topologically and algebraically, $\det \circ f$ is trivial by Theorem 2.3. Thus $\det(f(g)) = 1$ for all $g \in \text{SL}_2(\mathbb{R})$, so $f(\text{SL}_2(\mathbb{R})) \subset \text{SL}_n(\mathbb{R})$. \Box

**Example 2.5.** We will construct a continuous homomorphism $\text{GL}_2(\mathbb{R}) \to \text{GL}_3(\mathbb{R})$ and see its restriction to $\text{SL}_2(\mathbb{R})$ has values in $\text{SL}_3(\mathbb{R})$.

Let $V = \mathbb{R}x^2 + \mathbb{R}xy + \mathbb{R}y^2$ be the vector space of homogeneous polynomials in $x$ and $y$ of degree 2: quadratic forms on $\mathbb{R}^2$. This space is 3-dimensional, with basis $x^2, xy, y^2$. For $g = (a \ b \ c \ d)$ in $\text{GL}_2(\mathbb{R})$ and $Q(x, y)$ in $V$, set $(gQ)(x, y) = Q(ax + cy, bx + dy)$. If we think of quadratic forms and matrices acting on column vectors from the left, then $(gQ)(\begin{array}{c} x \\ y \end{array}) = Q(g^{-1} \begin{array}{c} x \\ y \end{array})$. Check that $g_1(g_2Q) = (g_1g_2)Q$, so $\text{GL}_2(\mathbb{R})$ acts on $V$ from the left.

For instance, let $Q(x, y) = x^2 + y^2$, $g_1 = (\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array})$, and $g_2 = (\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array})$. Then $g_1g_2 = (\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array})$, so

$$(g_1g_2)(x, y) = Q(x + y, x + y) = y^2 + (x + y)^2 = x^2 + 2xy + 2y^2,$$

$$(g_1)(g_2Q)(x, y) = (g_2Q)(x + y) = x^2 + 2x(x + y) + 2(x + y)^2 = 5x^2 + 6xy + 2y^2,$$

and

$$((g_1g_2)Q)(x, y) = Q(x + y, 2x + y) = (x + y)^2 + (2x + y)^2 = 5x^2 + 6xy + 2y^2,$$

which illustrates that $g_1(g_2Q) = (g_1g_2)Q$.

The left action of $\text{GL}_2(\mathbb{R})$ on $V$ above is a linear change of variables on $V$ given by the entries of $2 \times 2$ matrices. Since $g(Q + Q') = g(Q) + g(Q')$ and $g(sQ) = sg(Q)$ for $s \in \mathbb{R}$, the action of $g$ on $V$ is a linear transformation (necessarily invertible, since the action of $g^{-1}$ on $V$ is its inverse). Using the basis $x^2, xy, y^2$ of $V$, we can compute a matrix representation of $g$ on $V$: we have

$$g(x^2) = (ax + cy)^2 = a^2x^2 + 2acxy + c^2y^2,$$

$$g(xy) = (ax + cy)(bx + dy) = abx^2 + (ad + bc)xy + cdy^2,$$

2If $(g \ast Q)(x, y) = Q(ax + by, cx + dy)$, or equivalently $(g \ast Q)(\begin{array}{c} x \\ y \end{array}) = Q(g(\begin{array}{c} x \\ y \end{array}))$, then $g_1 \ast (g_2 \ast Q) = (g_2g_1) \ast Q.$
and
\[ g(y^2) = (bx + dy)^2 = b^2x^2 + 2bdxy + d^2y^2, \]
so the matrix of \( g \) with respect to the basis \( x^2, xy, y^2 \) is
\[
\begin{pmatrix}
  a^2 & ab & b^2 \\
  2ac & ad + bc & 2bd \\
  c^2 & cd & d^2
\end{pmatrix}.
\]
Call this matrix \( f(g) \), so \( f: \text{GL}_2(\mathbb{R}) \to \text{GL}_3(\mathbb{R}) \) is a homomorphism and the formula for \( f(g) \) shows \( f \) is continuous. By a calculation, \( \det(f(g)) = (ad - bc)^3 = (\det g)^3 \), so when \( g \) has determinant 1 so does \( f(g) \).

Example 2.5 can be generalized. For each integer \( n \geq 1 \), the space \( V_n = \bigoplus_{i=0}^n \mathbb{R}x^{n-i}y^i \) of homogeneous 2-variable polynomials of degree \( n \) has dimension \( n + 1 \) and \( \text{GL}_2(\mathbb{R}) \) acts on this space by linear changes of variables. The restriction of this action to \( \text{SL}_2(\mathbb{R}) \) on \( V_n \) accounts for essentially all “interesting” actions of \( \text{SL}_2(\mathbb{R}) \) on finite-dimensional vector spaces.

**Theorem 2.6.** The homomorphism \( \text{SL}_2(\mathbb{R}[x, y]) \to \text{SL}_2(\mathbb{R}[x, y]/(x^2 + y^2 - 1)) \), where matrix entries are reduced componentwise modulo \( x^2 + y^2 - 1 \), is not surjective.

**Proof.** Let \( \pi \) and \( \eta \) be the cosets of \( x \) and \( y \) in \( \mathbb{R}[x, y]/(x^2 + y^2 - 1) \), so \( \pi^2 + \eta^2 = 1 \). One matrix in \( \text{SL}_2(\mathbb{R}[x, y]/(x^2 + y^2 - 1)) \) is
\[
(2.5) \quad \begin{pmatrix} \pi & -\eta \\ \eta & \pi \end{pmatrix}.
\]
We will prove by contradiction that there is no matrix \( A(x, y) \) in \( \text{SL}_2(\mathbb{R}[x, y]) \) that becomes the matrix \( (2.5) \) when the entries of \( A(x, y) \) are reduced modulo \( x^2 + y^2 - 1 \).

For each matrix
\[
A(x, y) = \begin{pmatrix} a(x, y) & b(x, y) \\ c(x, y) & d(x, y) \end{pmatrix}
\]
in \( \text{SL}_2(\mathbb{R}[x, y]) \), we have \( \det A(x, y) = a(x, y)d(x, y) - b(x, y)c(x, y) = 1 \) in \( \mathbb{R}[x, y] \), so for all real numbers \( u \) and \( v \) we have \( A(u, v) \in \text{SL}_2(\mathbb{R}) \).

Suppose \( A(x, y) \) reduces to the matrix \( (2.5) \) in \( \text{SL}_2(\mathbb{R}[x, y]/(x^2 + y^2 - 1)) \). Then when \( u^2 + v^2 = 1 \) in \( \mathbb{R} \) we have \( A(u, v) = (u - v, \ u) \). Define the homotopy \( h_A: S^1 \times [0, 1] \to \text{SL}_2(\mathbb{R}) \) by \( h_A(u, v, t) = A(tu, tv) \). Then \( h_A(u, v, 0) = (u - v, \ u) \) is a loop in \( \text{SL}_2(\mathbb{R}) \) that generates \( \pi_1(\text{SL}_2(\mathbb{R})) \) (see Remark 2.2). Then \( h_A \) gives us a way to continuously shrink a generator of \( \pi_1(\text{SL}_2(\mathbb{R})) \) to a constant map. This is impossible since \( \pi_1(\text{SL}_2(\mathbb{R})) \) is nontrivial, so no matrix in \( \text{SL}_2(\mathbb{R}[x, y]) \) reduces to \( (2.5) \) modulo \( x^2 + y^2 - 1 \). \( \square \)

**Corollary 2.7.** The homomorphism \( \text{SL}_2(\mathbb{Z}[x, y]) \to \text{SL}_2(\mathbb{Z}[x, y]/(x^2 + y^2 - 1)) \), where matrix entries are reduced componentwise modulo \( x^2 + y^2 - 1 \), is not surjective.

**Proof.** The matrix \( (\pi - \eta, \pi) \) is in \( \text{SL}_2(\mathbb{Z}[x, y]/(x^2 + y^2 - 1)) \), and if some matrix \( A(x, y) \) in \( \mathbb{Z}[x, y] \) reduces to it then we can use \( A(x, y) \) in the proof of Theorem 2.6. \( \square \)

\(^3\)For \((u, v) \in S^1\), the matrix \( A(tu, tv) \) is not \((tu - tv, tu)\) if \( 0 \leq t < 1 \) since \( \det(tu - tv) = t^2 \neq 1 \).
3. Conjugacy Classes

The conjugacy class of a matrix in $SL_2(\mathbb{R})$ is nearly determined by its eigenvalues, but we have to be a little bit careful so we don’t confuse conjugacy in $SL_2(\mathbb{R})$ with conjugacy in the larger group $GL_2(\mathbb{R})$. For example, $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$ and its inverse $(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix})$ are conjugate in $GL_2(\mathbb{R})$ by $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$, whose determinant is $-1$. These two matrices are not conjugate in $SL_2(\mathbb{R})$, since each $SL_2(\mathbb{R})$-conjugate of $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$ has a positive upper right entry, by an explicit calculation.

**Theorem 3.1.** Let $T \in SL_2(\mathbb{R})$. If $(\text{Tr}T)^2 > 4$ then $T$ is conjugate to a unique matrix of the form $(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix})$ with $|\lambda| > 1$. If $(\text{Tr}T)^2 = 4$ then $T$ is conjugate to exactly one of $\pm I_2$, $\pm(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$, or $\pm(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$. If $(\text{Tr}T)^2 < 4$ then $T$ is conjugate to a unique matrix of the form $(\begin{smallmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{smallmatrix})$ other than $\pm I_2$.

*Proof.* For $T \in SL_2(\mathbb{R})$, its eigenvalues are roots of its characteristic polynomial, which is $X^2 - tX + 1$, where $t = \text{Tr}(T)$. The nature of the eigenvalues of $T$ are determined by the discriminant of this polynomial, $t^2 - 4$: two distinct real eigenvalues if $t^2 > 4$, a repeated eigenvalue if $t^2 = 4$, and two complex conjugate eigenvalues if $t^2 < 4$. We will find a representative for the conjugacy class of $T$ based on the sign of $t^2 - 4$. Of course matrices with different $t$’s are not conjugate.

In what follows, if $v$ and $w$ are vectors in $\mathbb{R}^2$ whose specific coordinates are not important to make explicit, we will write $([v] [w])$ for the matrix $(a b \\ c d)$. This matrix is invertible when $v$ and $w$ are linearly independent.

Suppose $t^2 > 4$. Then $T$ has distinct real eigenvalues $\lambda$ and $1/\lambda$. Let $v$ and $v'$ be eigenvectors in $\mathbb{R}^2$ for these eigenvalues: $Tv = \lambda v$ and $Tv' = (1/\lambda)v'$. In coordinates from the basis $v$ and $v'$, $T$ is represented by $(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix})$, so $T$ is conjugate to $(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix})$ by the $2 \times 2$ matrix $([v] [v'])$. Scaling $v'$ keeps it as an eigenvector of $T$, and by a suitable nonzero scaling the matrix $([v] [v'])$ has determinant $1$. Therefore $T$ is conjugate to $(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix})$ in $SL_2(\mathbb{R})$. We did not specify an ordering of the eigenvalues, so $(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix})$ and $(\begin{smallmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{smallmatrix})$ have to be conjugate to each other in $SL_2(\mathbb{R})$. Explicitly, $(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}) (\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix}) (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix})^{-1} = (\begin{smallmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{smallmatrix})$. Conjugate matrices have the same eigenvalues, so $(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix})$ is conjugate to $(\begin{smallmatrix} \mu & 0 \\ 0 & 1/\mu \end{smallmatrix})$ only when $\mu$ equals $\lambda$ or $1/\lambda$.

We can therefore pin down a representative for the conjugacy class of $T$ in $SL_2(\mathbb{R})$ as $(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix})$ with $|\lambda| > 1$.

Now suppose $t^2 = 4$. The roots of $X^2 - tX + 1$ are both $1$ (if $t = 2$) or both $-1$ (if $t = -2$). Let $\lambda = \pm 1$ be the eigenvalue for $T$. Extend $v$ to a basis $\{v, v'\}$ of $\mathbb{R}^2$. Scaling $v'$, we may assume the matrix $([v] [v'])$ has determinant $1$. Conjugating $T$ by this matrix expresses it in the basis $v$ and $v'$ as $(\begin{smallmatrix} \lambda & \mu \\ 0 & \nu \end{smallmatrix})$. Since the determinant is $1$, $y = 1/\lambda = \lambda = \pm 1$. Therefore $T$ is conjugate in $SL_2(\mathbb{R})$ to a matrix of the form $(\begin{smallmatrix} 1 & \mu \\ 0 & \nu \end{smallmatrix})$ or $(\begin{smallmatrix} -1 & \mu \\ 0 & \nu \end{smallmatrix})$.

If $x = 0$ these matrices are $\pm I_2$, which are in their own conjugacy class. The formulas
\[
\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}^{-1} = \begin{pmatrix} 1 & r^2 \\ 0 & 1 \end{pmatrix}
\]
and
\[
\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -r^2 \\ 0 & 1 \end{pmatrix}
\]
show $(\begin{smallmatrix} 1 & \mu \\ 0 & \nu \end{smallmatrix})$ is conjugate to either $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$ or $(\begin{smallmatrix} -1 & \mu \\ 0 & \nu \end{smallmatrix})$, depending on the sign of $x$. Similarly $(\begin{smallmatrix} -1 & \mu \\ 0 & \nu \end{smallmatrix})$ is conjugate to either $(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix})$ or $(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix})$.
The four matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ are nonconjugate in $SL_2(\mathbb{R})$, e.g., an $SL_2(\mathbb{R})$-conjugate of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ looks like $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ with $x$ a perfect square. Other cases are left to the reader.

Finally, suppose $t^2 < 4$. Now $T$ has complex conjugate eigenvalues that, by the quadratic formula for $X^2 - tX + 1$, are of absolute value 1 and are not $\pm 1$ (since $t \neq \pm 2$). We can write the eigenvalues as $e^{\pm i\theta}$, with $\sin \theta \neq 0$. Pick an eigenvector $v$ in $\mathbb{C}^2$ such that $Tv = e^{i\theta}v$. Since $e^{i\theta}$ is not real, $v \not\in \mathbb{R}^2$. Let $\overline{v}$ be the vector with coordinates that are complex conjugate to those of $v$, so $T\overline{v} = e^{-i\theta}\overline{v}$. Then $v + \overline{v}$ and $i(v - \overline{v})$ are in $\mathbb{R}^2$, with

$$T(v + \overline{v}) = (\cos \theta)(v + \overline{v}) + (\sin \theta)i(v - \overline{v})$$

and

$$T(v - \overline{v}) = -(\sin \theta)(v + \overline{v}) + (\cos \theta)i(v - \overline{v}).$$

Therefore conjugating $T$ by the (invertible) real matrix $[[v + \overline{v}] [i(v - \overline{v})]]$ turns $T$ into $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$. We don’t know the determinant of $[[v + \overline{v}] [i(v - \overline{v})]]$, but scaling $v$ by a real number (and $\overline{v}$ by the same amount, to keep it conjugate) can give this conjugating matrix determinant $\pm 1$. If the determinant is 1 then $T$ is conjugate to $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ in $SL_2(\mathbb{R})$. If the determinant is $-1$, then reverse the order of the columns in the conjugating matrix to give it determinant 1 and then $T$ is conjugate to $\begin{pmatrix} -\sin \theta & \cos \theta \\ \cos \theta & \sin \theta \end{pmatrix}$ in $SL_2(\mathbb{R})$. Two matrices $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ and $\begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}$ in $K$ can be conjugate only when $\varphi = \pm \theta \mod 2\pi \mathbb{Z}$, by looking at eigenvalues, and a direct calculation shows the $SL_2(\mathbb{R})$-conjugate of $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ never equals $\begin{pmatrix} \cos (-\theta) & \sin (-\theta) \\ \sin (-\theta) & \cos (-\theta) \end{pmatrix}$ unless $\sin \theta = 0$, but we are in a case when $\sin \theta \neq 0$. \hfill \Box

When $T \in SL_2(\mathbb{R})$ satisfies $\text{Tr}(T)^2 > 4$ we say $T$ is hyperbolic, when $(\text{Tr}T)^2 = 4$ we say $T$ is parabolic, and when $(\text{Tr}T)^2 < 4$ we say $T$ is elliptic. This terminology is borrowed from the shape of a plane conic $ax^2 + bxy + cy^2 = 1$ in terms of its discriminant $d = b^2 - 4ac$: it is a hyperbola when $d > 0$, a parabola when $d = 0$, and an ellipse when $d < 0$. Up to sign, the hyperbolic conjugacy classes in $SL_2(\mathbb{R})$ are represented by matrices in $A$ (besides $I_2$), the elliptic conjugacy classes are represented by matrices in $K$ (besides $\pm I_2$), and the parabolic conjugacy classes are represented by matrices in $N$.

**Appendix A. Acting on the Upper Half-Plane**

We will use an action of $SL_2(\mathbb{R})$ on the upper half-plane $\mathfrak{h} = \{x + iy : y > 0\}$ to obtain the Iwasawa decomposition of $SL_2(\mathbb{R})$ in a more efficient manner than the first proof that used an action on bases of $\mathbb{R}^2$.

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$ and a non-real complex number $z$, set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az + b}{cz + d} \in \mathbb{C} - \mathbb{R}. \tag{A.1}$$

By a calculation left to the reader, $g_1(g_2(z)) = (g_1g_2)(z)$ for $g_1$ and $g_2$ in $GL_2(\mathbb{R})$, and

$$\text{Im} \left( \frac{az + b}{cz + d} \right) = \frac{(ad - bc) \text{Im}(z)}{|cz + d|^2}.$$ 

Therefore when $ad - bc > 0$, $z$ and $(az + b)/(cz + d)$ have the same sign for their imaginary parts. In particular, if $g \in SL_2(\mathbb{R})$ and $z$ is in the upper half-plane then so is $g(z)$, so (A.1)
is an action of the group $\text{SL}_2(\mathbb{R})$ on the set $\mathfrak{h}$. This action has one orbit since we can get anywhere in $\mathfrak{h}$ from $i$ using $\text{SL}_2(\mathbb{R})$: 

(A.2) \[ \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} (i) = x + iy. \]

Notice that the matrix used here to send $i$ to $x + iy$ is in the subgroup $AN$ (see (2.4)).

Let's determine the stabilizer of $i$. Saying $(a b c d)(i) = i$ is equivalent to $(a i + b)/(c i + d) = i$, so $a i + b = d i - c$. Therefore $d = a$ and $b = -c$, so $(a b c d) = (\cos \theta \sin \theta \ - \sin \theta \cos \theta \ 0 \ 1)$ with $a^2 + c^2 = 1$. We can therefore write $a = \cos \theta$ and $c = \sin \theta$, which shows the stabilizer of $i$ is the set of matrices $(\cos \theta \ - \sin \theta \ 0 \ 1)$. This is the subgroup $K$ (so $\mathfrak{h}$ can be viewed as the coset space $\text{SL}_2(\mathbb{R})/K$ on which $\text{SL}_2(\mathbb{R})$ acts by left multiplication).

Now we are ready to derive the Iwasawa decomposition. For $g \in \text{SL}_2(\mathbb{R})$, write $g(i) = x + iy \in \mathfrak{h}$. Using (A.2),

(A.3) \[ g(i) = x + iy = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} (i) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} (i) = (NA)(i). \]

Since $g$ acts on $i$ in the same way as an element of $NA$, and the stabilizer of $i$ is $K$, $g \in NAK$. Thus $\text{SL}_2(\mathbb{R}) = NAK$. Applying inversion to this decomposition, $\text{SL}_2(\mathbb{R}) = KAN$. That settles the existence of the Iwasawa decomposition.

To prove uniqueness, assume $nak = n' a k'$. Applying both sides to $i$, $k$ and $k'$ fix $i$ so $na(i) = n' a'(i)$. For $n = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ and $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $na = \begin{pmatrix} r & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}$, so $na(i) = x + r^2 i$. In particular, knowing $na(i)$ tells us the parameters determining $n$ and $a$. Hence $n = n'$ and $a = a'$, so $k = k'$.

The upper half-plane action of $\text{SL}_2(\mathbb{R})$ leads in a second way to the formulas (2.1) and (2.2) for the matrix entries in the factors of the Iwasawa decomposition for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$. (We have just proved anew the existence and uniqueness of this decomposition.) Write, as in Section 2,

(A.4) \[ g = kan = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \]

We want to determine the entries of these matrices in terms of the entries of $g$. We will work with $g^{-1} = n^{-1} a^{-1} k^{-1}$ since the $\text{SL}_2(\mathbb{R})$-action on $\mathfrak{h}$ leads to the decomposition $NAK$ rather than $KAN$:

$g^{-1}(i) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} (i) = \frac{d i - b}{-ci + a} = \frac{a b c d}{a^2 + c^2} + \frac{1}{a^2 + c^2} i.$

Writing this as $u + iv$, from (A.3) (with $g^{-1}$ in place of $g$ and $u + iv$ in place of $x + iy$) we get

\[ n^{-1} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -(ab + cd)/(a^2 + c^2) \\ 0 & 1 \end{pmatrix} \]

and

\[ a^{-1} = \begin{pmatrix} \sqrt{v} & 0 \\ 0 & 1/\sqrt{v} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{a^2 + c^2} & c \\ 0 & \sqrt{a^2 + c^2} \end{pmatrix}, \]

so

\[ n = \begin{pmatrix} 1 & (ab + cd)/(a^2 + c^2) \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} \sqrt{a^2 + c^2} & 0 \\ 0 & 1/\sqrt{a^2 + c^2} \end{pmatrix}. \]
Since $g = kan$, 

$$k = gn^{-1}a^{-1} = \left( \begin{array}{cc} a/\sqrt{a^2 + c^2} & -c/\sqrt{a^2 + c^2} \\ c/\sqrt{a^2 + c^2} & a/\sqrt{a^2 + c^2} \end{array} \right).$$

The formulas for the entries of $k$, $a$, and $n$ match those in (2.1) and (2.2).

It is interesting to compare the role of the group $K$ in the geometry of $\mathbb{R}^2$ and $\mathfrak{h}$. As a transformation of $\mathbb{R}^2$, an element of $K$ is a rotation around the origin. This is an isometry of $\mathbb{R}^2$ using the Euclidean metric, and the $K$-orbit of a nonzero vector in $\mathbb{R}^2$ is the circle that passes through that vector and is centered at the origin. As a transformation of $\mathfrak{h}$, an element of $K$ is a rotation around $i$ relative to the hyperbolic metric on $\mathfrak{h}$. This is a hyperbolic isometry of $\mathfrak{h}$, and the $K$-orbit of a point in $\mathfrak{h}$ is the circle through that point that is centered at $i$ relative to the hyperbolic metric.

The conjugacy class of a matrix $T \in \text{SL}_2(\mathbb{R})$ was determined in Theorem 3.1 in terms of $(\text{Tr} T)^2 - 4$, which is the discriminant of the characteristic polynomial of $T$. The sign of this quantity tells us whether $T$ has real or non-real eigenvectors. The difference $(\text{Tr} T)^2 - 4$ is also relevant to the action of $T$ on the upper half-plane, with fixed points replacing eigenvectors. When $T = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$, the fixed-point condition $T(z) = z$ is equivalent to $az + b = (cz + d)z$, which says $cz^2 + (d - a)z - b = 0$. The discriminant of this equation, which tells us the number of real roots, is

$$(d - a)^2 + 4bc = d^2 - 2da + a^2 + 4(ad - 1) = (a + d)^2 - 4 = (\text{Tr} T)^2 - 4.$$