# DECOMPOSING $\mathrm{SL}_2(\mathbf{R})$

KEITH CONRAD

## 1. Introduction

The group $\mathrm{SL}_2(\mathbf{R})$ is not easy to visualize: it naturally lies in $\mathrm{M}_2(\mathbf{R})$, which is 4-dimensional (the entries of a variable $2 \times 2$ real matrix are 4 free parameters). We will derive a product decomposition for $\mathrm{SL}_2(\mathbf{R})$ and use it to get a concrete image of $\mathrm{SL}_2(\mathbf{R})$.

Inside $\mathrm{SL}_2(\mathbf{R})$ are the following three subgroups:

$$K = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}, \quad A = \left\{ \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} : r > 0 \right\}, \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}.$$

**Theorem 1.1.** *We have a decomposition* $\mathrm{SL}_2(\mathbf{R}) = KAN$*: every* $g \in \mathrm{SL}_2(\mathbf{R})$ *has a unique representation as* $g = kan$ *where* $k \in K$*,* $a \in A$*, and* $n \in N$*.*

This formula $\mathrm{SL}_2(\mathbf{R}) = KAN$ is called the *Iwasawa decomposition* of the group. Don't confuse the use of $a$ in Theorem 1.1 as the label for a matrix in $A$ with $a$ as a real number in the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The distinction should always be clear from the context. Since $\mathrm{SL}_2(\mathbf{R})$ is defined by the single equation $ad - bc = 1$ inside of $\mathrm{M}_2(\mathbf{R})$, it is a manifold of dimension $4 - 1 = 3$. The subgroups $K$, $A$, and $N$ are each 1-dimensional ($K \cong S^1$, $A \cong \mathbf{R}_{>0}$, and $N \cong \mathbf{R}$), and Theorem 1.1 shows they fully account for the 3 dimensions of $\mathrm{SL}_2(\mathbf{R})$.

The subgroups in the Iwasawa decomposition are related to conjugacy classes. We will see that a matrix in $\mathrm{SL}_2(\mathbf{R})$ is, up to sign, conjugate to a matrix in $K$, $A$, or $N$.

## 2. Iwasawa decomposition

To derive the Iwasawa decomposition of $\mathrm{SL}_2(\mathbf{R})$ we will use an action of this group on bases in $\mathbf{R}^2$.

For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{R})$, apply it to the standard basis $e_1, e_2$. The vectors

$$ge_1 = \begin{pmatrix} a \\ c \end{pmatrix}, \quad ge_2 = \begin{pmatrix} b \\ d \end{pmatrix}$$

are also a basis of $\mathbf{R}^2$. We will pass from this new basis of $\mathbf{R}^2$ back to the standard basis $e_1, e_2$ of $\mathbf{R}^2$ by a sequence of transformations in $\mathrm{SL}_2(\mathbf{R})$ that amounts to something like the Gram–Schmidt process (which turns a basis of $\mathbf{R}^n$ into an orthonormal basis of $\mathbf{R}^n$).

Let $\theta$ be the angle *from* the positive $x$-axis *to* $ge_1$. Let $\rho_\theta$ be the counterclockwise rotation of the plane around the origin by $\theta$, so $\rho_{-\theta}(ge_1)$ is on the positive $x$-axis. Because $\det g$ is positive, the ordered pair of vectors $(ge_1, ge_2)$ has the same orientation as the ordered pair $(e_1, e_2)$, so $\rho_{-\theta}(ge_2)$ is in the upper (rather than lower) half-plane.

Since $\rho_{-\theta}(ge_1)$ is a positive scalar multiple of $e_1$, we want to divide $\rho_{-\theta}(ge_1)$ by its length so it becomes $e_1$. Its length is $r = ||\rho_{-\theta}(ge_1)|| = ||ge_1|| = \sqrt{a^2 + c^2}$. Applying $\left(\begin{smallmatrix} 1/r & 0 \\ 0 & 1/r \end{smallmatrix}\right)$ will have the desired effect $\rho_{-\theta}(ge_1) \mapsto e_1$, but this matrix doesn't have determinant 1. On the other hand, $\left(\begin{smallmatrix} 1/r & 0 \\ 0 & r \end{smallmatrix}\right)$ also has the desired effect on $\rho_{-\theta}(ge_1)$ and has determinant 1. So apply the matrix $\left(\begin{smallmatrix} 1/r & 0 \\ 0 & r \end{smallmatrix}\right)$ to $\mathbf{R}^2$. It sends $\rho_{-\theta}(ge_1)$ to $\left(\begin{smallmatrix} 1/r & 0 \\ 0 & r \end{smallmatrix}\right)\rho_{-\theta}(ge_1) = e_1$. What does it do to $\rho_{-\theta}(ge_2)$? The vector $\left(\begin{smallmatrix} 1/r & 0 \\ 0 & r \end{smallmatrix}\right)\rho_{-\theta}(ge_2)$ is in the upper half-plane (because $\left(\begin{smallmatrix} 1/r & 0 \\ 0 & r \end{smallmatrix}\right)$ has positive determinant) and along with $e_1$ it forms two edges of a parallelogram with area 1 (because $\left(\begin{smallmatrix} 1/r & 0 \\ 0 & r \end{smallmatrix}\right)$ has determinant $\pm 1$). A parallelogram with area 1 having base $e_1$ must have height 1, so $\left(\begin{smallmatrix} 1/r & 0 \\ 0 & r \end{smallmatrix}\right)\rho_{-\theta}(ge_2) = \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)$ for some $x$.

Each horizontal shear transformation $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, which has determinant 1, fixes the $x$-axis and acts as a stretching along each horizontal line. Applying the horizontal shear transformation $\left(\begin{smallmatrix} 1 & -x \\ 0 & 1 \end{smallmatrix}\right)$ to $\mathbf{R}^2$ takes $\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = e_2$ and fixes $e_1$. We have finally returned to the standard basis $e_1, e_2$ from the basis $ge_1, ge_2$ by a sequence of transformations in $\mathrm{SL}_2(\mathbf{R})$. Our overall composite transformation is

$$\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix}\rho_{-\theta},$$

so

$$\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix}\rho_{-\theta}g$$

sends $e_1$ to $e_1$ and $e_2$ to $e_2$. A linear transformation on $\mathbf{R}^2$ is determined by what it does to a basis, so

$$\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix}\rho_{-\theta}g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Solving for $g$,

$$\begin{aligned} g &= \rho_\theta \begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix}^{-1}\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \\ &\in KAN. \end{aligned}$$

Such an expression for $g$ as a product $kan$ with $k \in K$, $a \in A$, and $n \in N$ is called the Iwasawa decomposition of $g$.

To check this decomposition is unique, for each angle $\theta$, $r > 0$, and $x \in \mathbf{R}$, set

$$g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r\cos\theta & xr\cos\theta - (1/r)\sin\theta \\ r\sin\theta & xr\sin\theta + (1/r)\cos\theta \end{pmatrix}.$$

If this is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$ then

$$r = \sqrt{a^2 + c^2} > 0, \quad \cos\theta = \frac{a}{r}, \quad \sin\theta = \frac{c}{r}, \tag{2.1}$$

and

$$x = \begin{cases} \frac{b + (1/r)\sin\theta}{r\cos\theta}, & \text{if } \cos\theta \neq 0, \\ \frac{d - (1/r)\cos\theta}{r\sin\theta}, & \text{if } \sin\theta \neq 0. \end{cases}$$

Substituting the formulas for $\cos\theta$ and $\sin\theta$ into the formula for $x$, and using $ad - bc = 1$, we obtain the uniform formula

$$x = \frac{ab + cd}{a^2 + c^2}. \tag{2.2}$$

All the parameters in the matrices making up the Iwasawa decomposition of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are determined in (2.1) and (2.2), so the Iwasawa decomposition is unique. This completes the proof of Theorem 1.1. In an appendix we derive the Iwasawa decomposition using a different action of $\mathrm{SL}_2(\mathbf{R})$, on the upper half-plane.

The Iwasawa decomposition for $\mathrm{SL}_2(\mathbf{R})$ extends to higher dimensions: $\mathrm{SL}_n(\mathbf{R}) = KAN$ where $K = \mathrm{SO}_n(\mathbf{R}) = \{T \in \mathrm{GL}_n(\mathbf{R}) : TT^\top = I_n, \det T = 1\}$, $A$ is the group of diagonal matrices with positive diagonal entries (and determinant 1) and $N$ is the group of upper-triangular matrices with 1's along the main diagonal. While $A$ and $N$ are both isomorphic to $\mathbf{R}$ when $n = 2$, $N$ becomes nonabelian for $n > 2$. The group $K$ is compact, the group $A \cong (\mathbf{R}_{>0})^{n-1} \cong \mathbf{R}^{n-1}$ is abelian, and $N$ is a nilpotent group.[1] This explains the notation $A$ and $N$, for abelian and nilpotent.

Returning to the case of $2 \times 2$ matrices, since

$$\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & r^2x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}, \tag{2.3}$$

we can move each element of $A$ past an element of $N$ (on either side) at the cost of changing the element of $N$. Therefore $AN = NA$ is a subgroup of $\mathrm{SL}_2(\mathbf{R})$. Explicitly,

$$AN = \left\{ \begin{pmatrix} y & x \\ 0 & 1/y \end{pmatrix} : y > 0, x \in \mathbf{R} \right\}. \tag{2.4}$$

The Iwasawa decomposition $KAN = K(AN)$ for $\mathrm{SL}_2(\mathbf{R})$ is the analogue of the polar decomposition $S^1 \times \mathbf{R}_{>0}$ for $\mathbf{C}^\times$.

In the Iwasawa decomposition, neither $K$ nor $AN$ (nor $A$ or $N$) is normal in $\mathrm{SL}_2(\mathbf{R})$. For example, the conjugate of an element of $K$ by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is usually not in $K$ and the conjugate of an element of $AN$ by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is usually not in $AN$. Because of the non-normality, it is not easy to describe the group operation in $\mathrm{SL}_2(\mathbf{R})$ in terms of its Iwasawa decomposition. This decomposition is important for other purposes, such as the following.

**Corollary 2.1.** *As a topological space,* $\mathrm{SL}_2(\mathbf{R})$ *is homeomorphic to the inside of a solid torus.*

*Proof.* Let $f\colon K \times A \times N \to \mathrm{SL}_2(\mathbf{R})$ by $f(k, a, n) = kan$. This is continuous, and by Theorem 1.1 it is surjective. We can write down an inverse function using the computations at the end of the proof of Theorem 1.1. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{R})$, define $r(g) = \sqrt{a^2 + c^2}$ and

$$k(g) = \begin{pmatrix} a/r(g) & -c/r(g) \\ c/r(g) & a/r(g) \end{pmatrix}, \quad a(g) = \begin{pmatrix} r(g) & 0 \\ 0 & 1/r(g) \end{pmatrix},$$

$$n(g) = \begin{pmatrix} 1 & (ab + cd)/(a^2 + c^2) \\ 0 & 1 \end{pmatrix}.$$

The function $g \mapsto (k(g), a(g), n(g))$ from $\mathrm{SL}_2(\mathbf{R})$ to $K \times A \times N$ is continuous and is an inverse to $f$.

Topologically, $K \cong S^1$, $A \cong \mathbf{R}_{>0} \cong \mathbf{R}$, and $N \cong \mathbf{R}$. Therefore topologically, $\mathrm{SL}_2(\mathbf{R}) \cong S^1 \times \mathbf{R}^2$. The plane $\mathbf{R}^2$ is homeomorphic to the open unit disc $D$ by $v \mapsto v/\sqrt{1 + ||v||^2}$

[1]The word nilpotent has different meanings in group theory and matrix theory. A group $G$ is called nilpotent if there is a finite tower of subgroups $\{e\} = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_r = G$ for some $r$ where $N_i \lhd N$ and $N_{i+1}/N_i \subset Z(G/N_i)$ for all $i$. That the subgroup $N$ of $\mathrm{GL}_n(\mathbf{R})$ is nilpotent is shown on pages 27 and 28 of `https://kconrad.math.uconn.edu/blurbs/grouptheory/subgpseries1.pdf`, where $N$ is written as $\mathrm{UT}_n(\mathbf{R})$. A square matrix is called nilpotent if it has a power equal to $O$, so invertible square matrices are never nilpotent in the matrix sense. Thus a subgroup of $\mathrm{GL}_n(\mathbf{R})$ might be nilpotent in the sense of group theory, but its elements are not nilpotent in the sense of matrix theory.

(with inverse $w \mapsto w/\sqrt{1-||w||^2}$), where $||\cdot||$ is the usual length function on $\mathbf{R}^2$, so as a topological space $\mathrm{SL}_2(\mathbf{R})$ is homeomorphic to $S^1 \times D$, which is the inside of a solid torus.

As an alternate ending, on the decomposition $K \times A \times N \cong S^1 \times \mathbf{R}_{>0} \times \mathbf{R}$ treat the product $\mathbf{R}_{>0} \times \mathbf{R}$ as the right half plane $\{x+iy : x > 0\}$ and identify it with the open unit disc $D$ by the Cayley transformation $z \mapsto (z-1)/(z+1)$. (Vertical lines in the half-plane are sent to circles inside $D$ that are tangent to the unit circle at 1.) □

Although the proof of Corollary 2.1 shows $\mathrm{SL}_2(\mathbf{R})$ and $S^1 \times \mathbf{R}^2$ are homeomorphic as topological spaces, they are not isomorphic as groups. Equivalently, the homeomorphism $K \times A \times N \to \mathrm{SL}_2(\mathbf{R})$ in Corollary 2.1 is not a group homomorphism.

The Iwasawa decomposition of a matrix in $K$, $A$, or $N$ is the obvious one. For a lower triangular matrix $\left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right)$, which is in none of these subgroups, the inverse map in the proof of Corollary 2.1 gives us the decomposition

$$\begin{pmatrix} 1/\sqrt{1+y^2} & -y/\sqrt{1+y^2} \\ y/\sqrt{1+y^2} & 1/\sqrt{1+y^2} \end{pmatrix} \begin{pmatrix} \sqrt{1+y^2} & 0 \\ 0 & 1/\sqrt{1+y^2} \end{pmatrix} \begin{pmatrix} 1 & y/(1+y^2) \\ 0 & 1 \end{pmatrix}.$$

**Remark 2.2.** The inside of a solid torus has a circle as a strong deformation retract, so the fundamental group of $\mathrm{SL}_2(\mathbf{R})$ is isomorphic to that of a circle: $\pi_1(\mathrm{SL}_2(\mathbf{R})) \cong \pi_1(S^1) \cong \mathbf{Z}$. From the connection between covering spaces and subgroups of the fundamental group, $\mathrm{SL}_2(\mathbf{R})$ admits a unique covering space of degree $d$ for each positive integer $d$ and the universal covering space of $\mathrm{SL}_2(\mathbf{R})$ is the inside of a solid cylinder $\mathbf{R} \times D$ (homeomorphic to $\mathbf{R}^3$). The degree-2 cover of $\mathrm{SL}(2,\mathbf{R})$ is an important group called the metaplectic group.

We can write down an explicit example of a noncontractible loop in $\mathrm{SL}_2(\mathbf{R})$: the subgroup $K$, or rather the obvious map $S^1 \to K$. To prove this loop is noncontractible in $\mathrm{SL}_2(\mathbf{R})$ we use the Iwasawa decomposition to write down a strong deformation retract from $\mathrm{SL}_2(\mathbf{R})$ to $K$. Let $h\colon \mathrm{SL}_2(\mathbf{R}) \times [0,1] \to K$ by

$$h(kan,t) = ka^t n^t = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r^t & 0 \\ 0 & 1/r^t \end{pmatrix} \begin{pmatrix} 1 & tx \\ 0 & 1 \end{pmatrix}.$$

This is continuous with $h(kan,0) = k$, $h(kan,1) = kan$, and $h(k,t) = k$. Therefore $\pi_1(\mathrm{SL}_2(\mathbf{R})) \cong \pi_1(K) \cong \mathbf{Z}$, so $K$ has to be a noncontractible loop in $\mathrm{SL}_2(\mathbf{R})$ since $K$ is noncontractible in $K$.

That the Iwasawa decomposition gives us a picture of $\mathrm{SL}_2(\mathbf{R})$ is a striking geometric application. Here is an algebraic application (whose punchline is the corollary).

**Theorem 2.3.** *The only continuous homomorphism* $\mathrm{SL}_2(\mathbf{R}) \to \mathbf{R}$ *is the trivial homomorphism.*

*Proof.* Let $f\colon \mathrm{SL}_2(\mathbf{R}) \to \mathbf{R}$ be a continuous homomorphism. Then

$$f(kan) = f(k) + f(a) + f(n).$$

We will show $f$ is trivial on $K$, $A$, and $N$, and thus $f$ is trivial on $KAN = \mathrm{SL}_2(\mathbf{R})$.

Since $K \cong S^1$, the elements of finite order in $K$ are dense. Since $\mathbf{R}$ has no elements of finite order except 0, $f$ is trivial on a dense subset of $K$ and thus is trivial on $K$ by continuity. (As an alternate argument, since $K$ is a compact group so is $f(K)$, and the only compact subgroup of $\mathbf{R}$ is $\{0\}$.)

Now we look at $f$ on $A$ and $N$. Since $A \cong \mathbf{R}_{>0} \cong \mathbf{R}$ by $\left(\begin{smallmatrix} r & 0 \\ 0 & 1/r \end{smallmatrix}\right) \mapsto \log r$ and $N \cong \mathbf{R}$ by $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mapsto x$, both algebraically and topologically, describing the continuous homomorphisms from $A$ and $N$ to $\mathbf{R}$ is the same as describing the continuous homomorphisms from $\mathbf{R}$ to

$\mathbf{R}$. All continuous homomorphisms $\mathbf{R} \to \mathbf{R}$ have the form $x \mapsto tx$ for some real number $t$ (see where 1 goes, call that $t$, and then appeal to the denseness of $\mathbf{Q}$ in $\mathbf{R}$). Therefore

$$f\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} = t\log r, \quad f\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = t'x$$

for some $t$ and $t'$. Applying $f$ to both sides of (2.3),

$$t\log r + t'x = t'r^2x + t\log r,$$

so $t'x = t'r^2x$ for all $r > 0$ and $x \in \mathbf{R}$. Thus $t' = 0$ (*e.g.*, take $x = 1$ and $r = 2$ to see this.) This shows $f$ is trivial on $N$.

It remains to show $f$ is trivial on $A$. For this we appeal to the conjugation relation

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}^{-1}$$

Applying $f$, we get $f\left(\begin{smallmatrix} r & 0 \\ 0 & 1/r \end{smallmatrix}\right) = -f\left(\begin{smallmatrix} r & 0 \\ 0 & 1/r \end{smallmatrix}\right)$, so $f\left(\begin{smallmatrix} r & 0 \\ 0 & 1/r \end{smallmatrix}\right) = 0$. □

**Corollary 2.4.** *Every continuous homomorphism* $\mathrm{SL}_2(\mathbf{R}) \to \mathrm{GL}_n(\mathbf{R})$ *has image in* $\mathrm{SL}_n(\mathbf{R})$.

*Proof.* Let $f\colon \mathrm{SL}_2(\mathbf{R}) \to \mathrm{GL}_n(\mathbf{R})$ be a continuous homomorphism. Composing $f$ with the determinant $\mathrm{GL}_n(\mathbf{R}) \to \mathbf{R}^\times$ gives a continuous homomorphism $\det \circ f\colon \mathrm{SL}_2(\mathbf{R}) \to \mathbf{R}^\times$. Since $\mathrm{SL}_2(\mathbf{R})$ is connected (Corollary 2.1), its image under $\det \circ f$ is a connected subgroup of $\mathbf{R}^\times$, so it lies in $\mathbf{R}_{>0}$. As $\mathbf{R}_{>0} \cong \mathbf{R}$ both topologically and algebraically, $\det \circ f$ is trivial by Theorem 2.3. Thus $\det(f(g)) = 1$ for all $g \in \mathrm{SL}_2(\mathbf{R})$, so $f(\mathrm{SL}_2(\mathbf{R})) \subset \mathrm{SL}_n(\mathbf{R})$. □

**Example 2.5.** We will construct a continuous homomorphism $\mathrm{GL}_2(\mathbf{R}) \to \mathrm{GL}_3(\mathbf{R})$ and see its restriction to $\mathrm{SL}_2(\mathbf{R})$ has values in $\mathrm{SL}_3(\mathbf{R})$.

Let $V = \mathbf{R}x^2 + \mathbf{R}xy + \mathbf{R}y^2$ be the vector space of homogeneous polynomials in $x$ and $y$ of degree 2: quadratic forms on $\mathbf{R}^2$. This space is 3-dimensional, with basis $x^2, xy, y^2$. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}_2(\mathbf{R})$ and $Q(x,y)$ in $V$, set $(gQ)(x,y) = Q(ax + cy, bx + dy)$. If we think of quadratic forms and matrices acting on column vectors from the left, then $(gQ)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = Q(g^\top\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right))$. Check that $g_1(g_2Q) = (g_1g_2)Q$, so $\mathrm{GL}_2(\mathbf{R})$ acts on $V$ from the left.

For instance, let $Q(x,y) = x^2 + y^2$, $g_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and $g_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Then $g_1g_2 = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$, so

$$(g_2Q)(x,y) = Q(y, x+y) = y^2 + (x+y)^2 = x^2 + 2xy + 2y^2,$$

$$(g_1(g_2Q))(x,y) = (g_2Q)(x, x+y) = x^2 + 2x(x+y) + 2(x+y)^2 = 5x^2 + 6xy + 2y^2,$$

and

$$((g_1g_2)Q)(x,y) = Q(x+y, 2x+y) = (x+y)^2 + (2x+y)^2 = 5x^2 + 6xy + 2y^2,$$

which illustrates that $g_1(g_2Q) = (g_1g_2)Q$.[2]

The left action of $\mathrm{GL}_2(\mathbf{R})$ on $V$ above is a linear change of variables on $V$ given by the entries of $2 \times 2$ matrices. Since $g(Q + Q') = g(Q) + g(Q')$ and $g(sQ) = sg(Q)$ for $s \in \mathbf{R}$, the action of $g$ on $V$ is a linear transformation (necessarily invertible, since the action of $g^{-1}$ on $V$ is its inverse). Using the basis $x^2, xy, y^2$ of $V$, we can compute a matrix representation of $g$ on $V$: we have

$$g(x^2) = (ax + cy)^2 = a^2x^2 + 2acxy + c^2y^2,$$
$$g(xy) = (ax + cy)(bx + dy) = abx^2 + (ad + bc)xy + cdy^2,$$

[2]If $(g * Q)(x,y) = Q(ax + by, cx + dy)$, or equivalently $(g * Q)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = Q(g\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right))$, then $g_1 * (g_2 * Q) = (g_2g_1) * Q$.

and

$$g(y^2) = (bx + dy)^2 = b^2x^2 + 2bdxy + d^2y^2,$$

so the matrix of $g$ with respect to the basis $x^2, xy, y^2$ is

$$\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

Call this matrix $f(g)$, so $f\colon \mathrm{GL}_2(\mathbf{R}) \to \mathrm{GL}_3(\mathbf{R})$ is a homomorphism and the formula for $f(g)$ shows $f$ is continuous. By a calculation, $\det(f(g)) = (ad - bc)^3 = (\det g)^3$, so when $g$ has determinant 1 so does $f(g)$.

Example 2.5 can be generalized. For each integer $n \geq 1$, the space $V_n = \bigoplus_{i=0}^n \mathbf{R}x^{n-i}y^i$ of homogeneous 2-variable polynomials of degree $n$ has dimension $n + 1$ and $\mathrm{GL}_2(\mathbf{R})$ acts on this space by linear changes of variables. The restriction of this action to $\mathrm{SL}_2(\mathbf{R})$ on $V_n$ accounts for essentially all "interesting" actions of $\mathrm{SL}_2(\mathbf{R})$ on finite-dimensional vector spaces.

**Theorem 2.6.** *The homomorphism* $\mathrm{SL}_2(\mathbf{R}[x, y]) \to \mathrm{SL}_2(\mathbf{R}[x, y]/(x^2+y^2-1))$*, where matrix entries are reduced componentwise modulo* $x^2 + y^2 - 1$*, is not surjective.*

*Proof.* Let $\overline{x}$ and $\overline{y}$ be the cosets of $x$ and $y$ in $\mathbf{R}[x, y]/(x^2 + y^2 - 1)$, so $\overline{x}^2 + \overline{y}^2 = 1$. One matrix in $\mathrm{SL}_2(\mathbf{R}[x, y]/(x^2 + y^2 - 1))$ is

$$\begin{pmatrix} \overline{x} & -\overline{y} \\ \overline{y} & \overline{x} \end{pmatrix}. \tag{2.5}$$

We will prove by contradiction that there is no matrix $A(x, y)$ in $\mathrm{SL}_2(\mathbf{R}[x, y])$ that becomes the matrix (2.5) when the entries of $A(x, y)$ are reduced modulo $x^2 + y^2 - 1$.

For each matrix

$$A(x, y) = \begin{pmatrix} a(x, y) & b(x, y) \\ c(x, y) & d(x, y) \end{pmatrix}$$

in $\mathrm{SL}_2(\mathbf{R}[x, y])$, we have $\det A(x, y) = a(x, y)d(x, y) - b(x, y)c(x, y) = 1$ in $\mathbf{R}[x, y]$, so for all real numbers $u$ and $v$ we have $A(u, v) \in \mathrm{SL}_2(\mathbf{R})$.

Suppose $A(x, y)$ reduces to the matrix (2.5) in $\mathrm{SL}_2(\mathbf{R}[x, y]/(x^2 + y^2 - 1))$. Then when $u^2 + v^2 = 1$ in $\mathbf{R}$ we have $A(u, v) = \left(\begin{smallmatrix} u & -v \\ v & u \end{smallmatrix}\right)$. Define the homotopy $h_A\colon S^1 \times [0, 1] \to \mathrm{SL}_2(\mathbf{R})$ by $h_A(u, v, t) = A(tu, tv)$.[3] Then $h_A(u, v, 0)$ is a constant map while $h_A(u, v, 1) = A(u, v) = \left(\begin{smallmatrix} u & -v \\ v & u \end{smallmatrix}\right)$ is a loop in $\mathrm{SL}_2(\mathbf{R})$ that generates $\pi_1(\mathrm{SL}_2(\mathbf{R}))$ (see Remark 2.2). Then $h_A$ gives us a way to continuously shrink a generator of $\pi_1(\mathrm{SL}_2(\mathbf{R}))$ to a constant map. This is impossible since $\pi_1(\mathrm{SL}_2(\mathbf{R}))$ is nontrivial, so no matrix in $\mathrm{SL}_2(\mathbf{R}[x, y])$ reduces to (2.5) modulo $x^2 + y^2 - 1$. □

**Corollary 2.7.** *The homomorphism* $\mathrm{SL}_2(\mathbf{Z}[x, y]) \to \mathrm{SL}_2(\mathbf{Z}[x, y]/(x^2+y^2-1))$*, where matrix entries are reduced componentwise modulo* $x^2 + y^2 - 1$*, is not surjective.*

*Proof.* The matrix $\left(\begin{smallmatrix} \overline{x} & -\overline{y} \\ \overline{y} & \overline{x} \end{smallmatrix}\right)$ is in $\mathrm{SL}_2(\mathbf{Z}[x, y]/(x^2 + y^2 - 1))$, and if some matrix $A(x, y)$ in $\mathbf{Z}[x, y]$ reduces to it then we can use $A(x, y)$ in the proof of Theorem 2.6. □

[3] For $(u, v) \in S^1$, the matrix $A(tu, tv)$ is *not* $\left(\begin{smallmatrix} tu & -tv \\ tv & tu \end{smallmatrix}\right)$ if $0 \leq t < 1$ since $\det\left(\begin{smallmatrix} tu & -tv \\ tv & tu \end{smallmatrix}\right) = t^2 \neq 1$.

## 3. Conjugacy Classes

The conjugacy class of a matrix in $\mathrm{SL}_2(\mathbf{R})$ is nearly determined by its eigenvalues, but we have to be a little bit careful so we don't confuse conjugacy in $\mathrm{SL}_2(\mathbf{R})$ with conjugacy in the larger group $\mathrm{GL}_2(\mathbf{R})$. For example, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and its inverse $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ are conjugate in $\mathrm{GL}_2(\mathbf{R})$ by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, whose determinant is $-1$. These two matrices are not conjugate in $\mathrm{SL}_2(\mathbf{R})$, since each $\mathrm{SL}_2(\mathbf{R})$-conjugate of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ has a positive upper right entry, by an explicit calculation.

**Theorem 3.1.** *Let $T \in \mathrm{SL}_2(\mathbf{R})$. If $(\mathrm{Tr}\, T)^2 > 4$ then $T$ is conjugate to a unique matrix of the form $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix}\right)$ with $|\lambda| > 1$. If $(\mathrm{Tr}\, T)^2 = 4$ then $T$ is conjugate to exactly one of $\pm I_2$, $\pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, or $\pm\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$. If $(\mathrm{Tr}\, T)^2 < 4$ then $T$ is conjugate to a unique matrix of the form $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ other than $\pm I_2$.*

*Proof.* For $T \in \mathrm{SL}_2(\mathbf{R})$, its eigenvalues are roots of its characteristic polynomial, which is $X^2 - tX + 1$, where $t = \mathrm{Tr}(T)$. The nature of the eigenvalues of $T$ are determined by the discriminant of this polynomial, $t^2 - 4$: two distinct real eigenvalues if $t^2 > 4$, a repeated eigenvalue if $t^2 = 4$, and two complex conjugate eigenvalues if $t^2 < 4$. We will find a representative for the conjugacy class of $T$ based on the sign of $t^2 - 4$. Of course matrices with different $t$'s are not conjugate.

In what follows, if $v$ and $w$ are vectors in $\mathbf{R}^2$ whose specific coordinates are not important to make explicit, we will write $([v]\ [w])$ for the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. This matrix is invertible when $v$ and $w$ are linearly independent.

Suppose $t^2 > 4$. Then $T$ has distinct real eigenvalues $\lambda$ and $1/\lambda$. Let $v$ and $v'$ be eigenvectors in $\mathbf{R}^2$ for these eigenvalues: $Tv = \lambda v$ and $Tv' = (1/\lambda)v'$. In coordinates from the basis $v$ and $v'$, $T$ is represented by $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix}\right)$, so $T$ is conjugate to $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix}\right)$ by the $2 \times 2$ matrix $([v]\ [v'])$. Scaling $v'$ keeps it as an eigenvector of $T$, and by a suitable nonzero scaling the matrix $([v]\ [v'])$ has determinant 1. Therefore $T$ is conjugate to $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{R})$. We did not specify an ordering of the eigenvalues, so $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ have to be conjugate to each other in $\mathrm{SL}_2(\mathbf{R})$. Explicitly, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$. Conjugate matrices have the same eigenvalues, so $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix}\right)$ is conjugate to $\left(\begin{smallmatrix} \mu & 0 \\ 0 & 1/\mu \end{smallmatrix}\right)$ only when $\mu$ equals $\lambda$ or $1/\lambda$. We can therefore pin down a representative for the conjugacy class of $T$ in $\mathrm{SL}_2(\mathbf{R})$ as $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix}\right)$ with $|\lambda| > 1$.

Now suppose $t^2 = 4$. The roots of $X^2 - tX + 1$ are both 1 (if $t = 2$) or both $-1$ (if $t = -2$). Let $\lambda = \pm 1$ be the eigenvalue for $T$. Extend $v$ to a basis $\{v, v'\}$ of $\mathbf{R}^2$. Scaling $v'$, we may assume the matrix $([v]\ [v'])$ has determinant 1. Conjugating $T$ by this matrix expresses it in the basis $v$ and $v'$ as $\left(\begin{smallmatrix} \lambda & x \\ 0 & y \end{smallmatrix}\right)$. Since the determinant is 1, $y = 1/\lambda = \lambda = \pm 1$. Therefore $T$ is conjugate in $\mathrm{SL}_2(\mathbf{R})$ to a matrix of the form $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} -1 & x \\ 0 & -1 \end{smallmatrix}\right)$. If $x = 0$ these matrices are $\pm I_2$, which are in their own conjugacy class. The formulas

$$\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}^{-1} = \begin{pmatrix} 1 & r^2 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -r^2 \\ 0 & 1 \end{pmatrix}$$

show $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ is conjugate to either $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, depending on the sign of $x$. Similarly $\left(\begin{smallmatrix} -1 & x \\ 0 & -1 \end{smallmatrix}\right)$ is conjugate to either $\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$.

The four matrices $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$ are nonconjugate in $\mathrm{SL}_2(\mathbf{R})$, *e.g.*, an $\mathrm{SL}_2(\mathbf{R})$-conjugate of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ looks like $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ with $x$ a perfect square. Other cases are left to the reader.

Finally, suppose $t^2 < 4$. Now $T$ has complex conjugate eigenvalues that, by the quadratic formula for $X^2 - tX + 1$, are of absolute value 1 and are not $\pm 1$ (since $t \neq \pm 2$). We can write the eigenvalues as $e^{\pm i\theta}$, with $\sin\theta \neq 0$. Pick an eigenvector $v$ in $\mathbf{C}^2$ such that $Tv = e^{i\theta}v$. Since $e^{i\theta}$ is not real, $v \notin \mathbf{R}^2$. Let $\overline{v}$ be the vector with coordinates that are complex conjugate to those of $v$, so $T\overline{v} = e^{-i\theta}\overline{v}$. Then $v + \overline{v}$ and $i(v - \overline{v})$ are in $\mathbf{R}^2$, with

$$T(v + \overline{v}) = (\cos\theta)(v + \overline{v}) + (\sin\theta)i(v - \overline{v})$$

and

$$T(v - \overline{v}) = -(\sin\theta)(v + \overline{v}) + (\cos\theta)i(v - \overline{v}).$$

Therefore conjugating $T$ by the (invertible) real matrix $([v + \overline{v}]\ [i(v - \overline{v})])$ turns $T$ into $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$. We don't know the determinant of $([v + \overline{v}]\ [i(v - \overline{v})])$, but scaling $v$ by a real number (and $\overline{v}$ by the same amount, to keep it conjugate) can give this conjugating matrix determinant $\pm 1$. If the determinant is 1 then $T$ is conjugate to $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{R})$. If the determinant is $-1$, then reverse the order of the columns in the conjugating marix to give it determinant 1 and then $T$ is conjugate to $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) = \left(\begin{smallmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{R})$. Two matrices $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{smallmatrix}\right)$ in $K$ can be conjugate only when $\varphi = \pm\theta \bmod 2\pi\mathbf{Z}$, by looking at eigenvalues, and a direct calculation shows the $\mathrm{SL}_2(\mathbf{R})$-conjugate of $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ never equals $\left(\begin{smallmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{smallmatrix}\right)$ unless $\sin\theta = 0$, but we are in a case when $\sin\theta \neq 0$. $\square$

When $T \in \mathrm{SL}_2(\mathbf{R})$ satisfies $\mathrm{Tr}(T)^2 > 4$ we say $T$ is *hyperbolic*, when $(\mathrm{Tr}\,T)^2 = 4$ we say $T$ is *parabolic*, and when $(\mathrm{Tr}\,T)^2 < 4$ we say $T$ is *elliptic*. This terminology is borrowed from the shape of a plane conic $ax^2 + bxy + cy^2 = 1$ in terms of its discriminant $d = b^2 - 4ac$: it is a hyperbola when $d > 0$, a parabola when $d = 0$, and an ellipse when $d < 0$. Up to sign, the hyperbolic conjugacy classes in $\mathrm{SL}_2(\mathbf{R})$ are represented by matrices in $A$ (besides $I_2$), the elliptic conjugacy classes are represented by matrices in $K$ (besides $\pm I_2$), and the parabolic conjugacy classes are represented by matrices in $N$.

## Appendix A. Acting on the Upper Half-Plane

We will use an action of $\mathrm{SL}_2(\mathbf{R})$ on the upper half-plane $\mathfrak{h} = \{x + iy : y > 0\}$ to obtain the Iwasawa decomposition of $\mathrm{SL}_2(\mathbf{R})$ in a more efficient manner than the first proof that used an action on bases of $\mathbf{R}^2$.

For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{R})$ and a non-real complex number $z$, set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az + b}{cz + d} \in \mathbf{C} - \mathbf{R}. \tag{A.1}$$

By a calculation left to the reader, $g_1(g_2(z)) = (g_1g_2)(z)$ for $g_1$ and $g_2$ in $\mathrm{GL}_2(\mathbf{R})$, and

$$\mathrm{Im}\left(\frac{az + b}{cz + d}\right) = \frac{(ad - bc)\,\mathrm{Im}(z)}{|cz + d|^2}.$$

Therefore when $ad - bc > 0$, $z$ and $(az + b)/(cz + d)$ have the same sign for their imaginary parts. In particular, if $g \in \mathrm{SL}_2(\mathbf{R})$ and $z$ is in the upper half-plane then so is $g(z)$, so (A.1) is an action of the group $\mathrm{SL}_2(\mathbf{R})$ on the set $\mathfrak{h}$. This action has one orbit since we can get anywhere in $\mathfrak{h}$ from $i$ using $\mathrm{SL}_2(\mathbf{R})$:

$$\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}(i) = x + iy. \tag{A.2}$$

Notice that the matrix used here to send $i$ to $x+iy$ is in the subgroup $AN$ (see (2.4)).

Let's determine the stabilizer of $i$. Saying $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})(i) = i$ is equivalent to $(ai+b)/(ci+d) = i$, so $ai + b = di - c$. Therefore $d = a$ and $b = -c$, so $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) = (\begin{smallmatrix} a & -c \\ c & a \end{smallmatrix})$ with $a^2 + c^2 = 1$. We can therefore write $a = \cos\theta$ and $c = \sin\theta$, which shows the stabilizer of $i$ is the set of matrices $(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix})$. This is the subgroup $K$ (so $\mathfrak{h}$ can be viewed as the coset space $\mathrm{SL}_2(\mathbf{R})/K$ on which $\mathrm{SL}_2(\mathbf{R})$ acts by left multiplication).

Now we are ready to derive the Iwasawa decomposition. For $g \in \mathrm{SL}_2(\mathbf{R})$, write $g(i) = x + iy \in \mathfrak{h}$. Using (A.2),

$$g(i) = x + iy = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}(i) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}(i) \in NA(i). \tag{A.3}$$

Since $g$ acts on $i$ in the same way as an element of $NA$, and the stabilizer of $i$ is $K$, $g \in NAK$. Thus $\mathrm{SL}_2(\mathbf{R}) = NAK$. Applying inversion to this decomposition, $\mathrm{SL}_2(\mathbf{R}) = KAN$. That settles the existence of the Iwasawa decomposition.

To prove uniqueness, assume $nak = n'a'k'$. Applying both sides to $i$, $k$ and $k'$ fix $i$ so $na(i) = n'a'(i)$. For $n = (\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix})$ and $a = (\begin{smallmatrix} r & 0 \\ 0 & 1/r \end{smallmatrix})$, $na = (\begin{smallmatrix} r & x/r \\ 0 & 1/r \end{smallmatrix})$, so $na(i) = x + r^2 i$. In particular, knowing $na(i)$ tells us the parameters determining $n$ and $a$. Hence $n = n'$ and $a = a'$, so $k = k'$.

The upper half-plane action of $\mathrm{SL}_2(\mathbf{R})$ leads in a second way to the formulas (2.1) and (2.2) for the matrix entries in the factors of the Iwasawa decomposition for $g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{SL}_2(\mathbf{R})$. (We have just proved anew the existence and uniqueness of this decomposition.) Write, as in Section 2,

$$g = kan = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \tag{A.4}$$

We want to determine the entries of these matrices in terms of the entries of $g$. We will work with $g^{-1} = n^{-1}a^{-1}k^{-1}$ since the $\mathrm{SL}_2(\mathbf{R})$-action on $\mathfrak{h}$ leads to the decomposition $NAK$ rather than $KAN$:

$$g^{-1}(i) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}(i) = \frac{di - b}{-ci + a} = -\frac{ab + cd}{a^2 + c^2} + \frac{1}{a^2 + c^2} i.$$

Writing this as $u + iv$, from (A.3) (with $g^{-1}$ in place of $g$ and $u + iv$ in place of $x + iy$) we get

$$n^{-1} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -(ab + cd)/(a^2 + c^2) \\ 0 & 1 \end{pmatrix}$$

and

$$a^{-1} = \begin{pmatrix} \sqrt{v} & 0 \\ 0 & 1/\sqrt{v} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{a^2 + c^2} & 0 \\ 0 & \sqrt{a^2 + c^2} \end{pmatrix},$$

so

$$n = \begin{pmatrix} 1 & (ab + cd)/(a^2 + c^2) \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} \sqrt{a^2 + c^2} & 0 \\ 0 & 1/\sqrt{a^2 + c^2} \end{pmatrix}.$$

Since $g = kan$,

$$k = gn^{-1}a^{-1} = \begin{pmatrix} a/\sqrt{a^2+c^2} & -c/\sqrt{a^2+c^2} \\ c/\sqrt{a^2+c^2} & a/\sqrt{a^2+c^2} \end{pmatrix}.$$

The formulas for the entries of $k$, $a$, and $n$ match those in (2.1) and (2.2).

It is interesting to compare the role of the group $K$ in the geometry of $\mathbf{R}^2$ and $\mathfrak{h}$. As a transformation of $\mathbf{R}^2$, an element of $K$ is a rotation around the origin. This is an isometry of $\mathbf{R}^2$ using the Euclidean metric, and the $K$-orbit of a nonzero vector in $\mathbf{R}^2$ is the circle that passes through that vector and is centered at the origin. As a transformation of $\mathfrak{h}$, an element of $K$ is a rotation around $i$ relative to the hyperbolic metric on $\mathfrak{h}$. This is a hyperbolic isometry of $\mathfrak{h}$, and the $K$-orbit of a point in $\mathfrak{h}$ is the circle through that point that is centered at $i$ relative to the hyperbolic metric.

The conjugacy class of a matrix $T \in \mathrm{SL}_2(\mathbf{R})$ was determined in Theorem 3.1 in terms of $(\operatorname{Tr} T)^2 - 4$, which is the discriminant of the characteristic polynomial of $T$. The sign of this quantity tells us whether $T$ has real or non-real eigenvectors. The difference $(\operatorname{Tr} T)^2 - 4$ is also relevant to the action of $T$ on the upper half-plane, with fixed points replacing eigenvectors. When $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the fixed-point condition $T(z) = z$ is equivalent to $az + b = (cz + d)z$, which says $cz^2 + (d - a)z - b = 0$. The discriminant of this equation, which tells us the number of real roots, is

$$(d-a)^2 + 4bc = d^2 - 2da + a^2 + 4(ad-1) = (a+d)^2 - 4 = (\operatorname{Tr} T)^2 - 4.$$